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# A Model for Estimating the Probability of Collision During the Execution of an In-Trail Procedure

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## 1 Introduction and scenario

1.1 When two airplanes are flying in the same direction, at different flight levels of the same oceanic route, it sometimes happens that one of them needs to climb or descend through the flight level of the other. For example, the airplane at the lower flight level may find that it has burned enough fuel to enable it to fly more efficiently at a higher altitude. It could also happen that one of the pilots needs to change his flight level in order to avoid turbulence.

1.2 If the longitudinal separation between the airplanes exceeds the minimum distance or minimum time specified for aircraft assigned to the same route and flight level, then the responsible air traffic controller can approve a request for a change of altitude without any concern that the two airplanes will have insufficient longitudinal separation during the period in which they have vertically overlapping positions. In recent years it has also been suggested that such altitude changes should be permitted with a smaller minimum longitudinal separation, since the period of vertical overlap is relatively brief when one airplane climbs or descends through another's flight level – and thus the exposure to risk is also relatively brief. The sequence of events by which such a climb or descent would be accomplished is typically known as an "in-trail procedure", or ITP. This report derives a formula for estimating the probability that a collision occurs when an ITP is executed.

1.3 Suppose that airplanes $a_1$ and $a_2$ are traveling in the same direction, on different flight levels of the same route; and suppose that $a_2$ requests a climb or descent through the flight level of $a_1$, which is not expected to change its altitude. We model the airplanes as rectangular solids of length $l$, width $w$, and height $h$. Let $m$ denote the minimum allowable longitudinal separation between airplanes assigned to the same route and flight level. Let $k$ be a number strictly between 0 and 1; and assume that an ITP may be authorized as long as the longitudinal distance between the airplanes, when $a_2$ begins its climb or descent, is expected to exceed $km$. (Some proposals have suggested using $k = ½$. Values of $k$ close to 1 yield conservative procedures; those close to 0 involve greater risk of collision.) Also assume that the responsible controller – using whatever manual or automated tools may be available to him – estimates that the longitudinal separation between $a_1$ and $a_2$ will be between $km$ and $m$ at the moment when $a_2$ begins its (requested) climb or descent. Finally, assume that the controller takes advantage of a rule that allows an altitude change when the longitudinal separation exceeds $km$, and authorizes the requested change.

1.4 In estimating the probability that $a_1$ and $a_2$ collide, we follow an approach used in the well-known Reich collision risk models, and assume that airplanes can collide in only one of three possible ways: nose-to-tail, top-to-bottom, or side-to-side. A nose-to-tail collision occurs if and only if the airplanes enter into longitudinal overlap during a period in which they are simultaneously in lateral and vertical overlap. A top-to-bottom collision occurs if and only if the airplanes enter into vertical overlap during a period in which they are simultaneously in lateral and longitudinal overlap. A side-to-side collision occurs if and only if the airplanes enter into lateral overlap during a period in which they are simultaneously in longitudinal and vertical overlap.

1.5 We generally denote constants by lower-case letters, and random variables by upper-case letters. $\varphi$ denotes the standard normal density function, $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. $\Phi$ denotes the standard normal distribution function, $\Phi(x) = \int_{-\infty}^{x} \varphi(u)\,du$. We let $\Psi(x) = \int_{-\infty}^{x} \Phi(u)\,du$ denote the integral of the standard normal distribution function.

## 2 Conditions for simultaneous longitudinal and vertical overlap

2.1 Since $a_1$ and $a_2$ are initially assigned to different flight levels, there is no reason to expect their speeds to be dependent. We assume their speeds, $V_1$ and $V_2$ respectively, to be independent random variables having the same normal (Gaussian) distribution. (Though speed varies from one airplane to another, each airplane's speed is assumed to be constant during the time period in which $a_2$ is changing its altitude.) We let $V = V_2 - V_1$ denote the signed difference between the speeds. $V$ is positive when $a_2$ is faster than $a_1$, and negative when $a_2$ is slower than $a_1$. Since $V_1$ and $V_2$ have the same normal distribution, their difference, $V$, is also normally distributed, and has mean 0. Let $\sigma^2$ denote the variance of $V$.

2.2 We set the origin of a time axis at the moment when $a_2$ begins its climb or descent, and let $U$ denote the signed initial distance by which (the center of) $a_2$ is ahead of (the center of) $a_1$. Thus $U$ is positive if $a_2$ is ahead of $a_1$ at $t = 0$, and negative if $a_2$ is behind $a_1$ at that time. At any particular time $t$ after the beginning of the climb or descent, the signed distance by which $a_2$ is ahead of $a_1$ is $U + Vt$. Lacking any reason to think otherwise, we assume that $U$ and $V$ are independent.

2.3 If the route system always operated perfectly, we would expect $U$ to be uniformly distributed on $(-m, -km] \cup [km, m)$. Recognizing that equipment occasionally fails and people sometimes make mistakes, we more realistically assume that there is a small probability $b > 0$ that the value of $U$ falls in the interval $(-km, km)$. Since such events typically result from blunders, we conservatively assume that when $U$ falls in the interval $(-km, km)$ it is uniformly distributed on that interval. Thus $f_U$, the probability density function of $U$, is given by

$$f_U(u) = \begin{cases} 0 & \text{if } u \le -m \\ \dfrac{1-b}{2m(1-k)} & \text{if } -m < u \le -km \\ \dfrac{b}{2km} & \text{if } -km < u < km \\ \dfrac{1-b}{2m(1-k)} & \text{if } km \le u < m \\ 0 & \text{if } m \le u. \end{cases} \qquad (1)$$

2.4 We define six events by which we describe the values assumed by the pair of random variables $U$ and $V$:

$E_1$: $-m < U < -l$ and $V < 0$; $E_2$: $-m < U < -l$ and $V > 0$;
$E_3$: $-l \le U \le l$ and $V < 0$; $E_4$: $-l \le U \le l$ and $V > 0$;
$E_5$: $l < U < m$ and $V < 0$; and $E_6$: $l < U < m$ and $V > 0$.

The union of these six events is the set of all possible values of the ordered pair $(U, V)$, except those whose second coordinate is 0, i.e., except those points that lie on the horizontal axis of a cartesian plane. Since – as a subset of the plane – the axis has measure 0, there is no risk of omitting events of positive probability by omitting consideration of those values of $(U, V)$ for which $V = 0$.

2.5 We let $T_b$ and $T_e$ respectively denote the beginning and ending times of the interval in which the airplanes are in a state of longitudinal overlap. That is, $T_b$ is the earliest time $t$ for which $U + Vt$ is in the interval $[-l, l]$, and $T_e$ is the latest such time.

2.6 If the airplanes are in longitudinal overlap when $a_2$ begins its altitude change, i.e., if $-l \le U \le l$, then $T_b = 0$. If $V < 0$, so that $E_3$ occurs, then the overlap period ends when $U + V T_e = -l$; and thus $T_e = \dfrac{-l-U}{V}$. If $V > 0$, so that $E_4$ occurs, then the overlap period ends when $U + V T_e = l$; and thus $T_e = \dfrac{l-U}{V}$.

2.7 $E_2$ occurs if and only if $-m < U < -l$ and $V > 0$, i.e., if and only if $a_2$ is initially behind $a_1$, but flying faster than $a_1$. Then $U + V T_b = -l$, so that $T_b = \dfrac{-l-U}{V}$; and $U + V T_e = l$, so that $T_e = \dfrac{l-U}{V}$.

2.8 $E_5$ occurs if and only if $l < U < m$ and $V < 0$, i.e., if and only if $a_2$ is initially ahead of $a_1$, but flying slower than $a_1$. Then $U + V T_b = l$, so that $T_b = \dfrac{l-U}{V}$; and $U + V T_e = -l$, so that $T_e = \dfrac{-l-U}{V}$.

2.9 If $-m < U < -l$ and $V < 0$, i.e., if $E_1$ occurs, then $a_2$ is initially behind $a_1$, and continues to fall farther behind. If $l < U < m$ and $V > 0$, i.e., if $E_6$ occurs, then $a_2$ is initially ahead of $a_1$, and continues to move farther ahead. In both of these cases the airplanes fail to experience a longitudinal overlap.

2.10 Let $a$ denote the absolute value of the initial altitude difference between $a_1$ and $a_2$; and let $c$ denote the absolute value of $a_2$'s speed of climb or descent. Ideally we would like to treat

$a$ and $c$ as random variables; but since there are only a few values that need to be considered, we take the easier approach of treating them as constants, and redoing the computations for relevant values. If $a_2$ is climbing, then at any particular time $t$ during the climb, the signed distance by which $a_1$ is above $a_2$ is $a - ct$; and if $a_2$ is descending, then at any particular time $t$ during the descent, the signed distance by which $a_2$ is above $a_1$ is also $a - ct$. In either case the airplanes enter into vertical overlap at the time $t_b$ for which $a - ct_b = h$, so that $t_b = (a - h)/c$; and since the overlap lasts $2h/c$, it ends at time $t_e = t_b + 2h/c$. That is, the interval of vertical overlap, $[t_b, t_e]$, is $\left[\frac{a-h}{c}, \frac{a+h}{c}\right]$.

2.11 In order to avoid unnecessary mathematical complications we explicitly assume a relationship that is virtually certain to be satisfied in practice if ITPs are ever authorized. The initial vertical separation $a$, between $a_1$ and $a_2$, is likely to be 2,000 ft or 3,000 ft – surely no more 6,000 ft – which is almost 1 nmi. The ratio $l/h$, i.e., the ratio of an airplane's length to its height, is approximately 3. Therefore, the maximum value of $al/h$ is no greater than 3 nmi. In discussions of the possible values of the minimum initial longitudinal separation needed to conduct an ITP, i.e., in discussions of the possible values for $km$, the most frequently suggested value of $m$ is 30 nmi (the smallest longitudinal separation applicable in oceanic airspace), and the most frequently suggested value of $km$ is 15 nmi. While it is, of course, conceivable that a somewhat smaller value of $km$ could be adopted, it is not realistic to imagine that it would be very much smaller. We can – and do – safely assume that $km > al/h$.

## 3 The probability of simultaneous longitudinal and vertical overlap

3.1 Let $f_{U,V}$ denote the joint density function of $U$ and $V$. Since these two random variables are independent, their joint density function must be the product of their individual density functions. Since $U$ has the density given by equation (1), and $V$ has the normal distribution with mean 0 and variance $\sigma^2$, it immediately follows that

$$f_{U,V}(u,v) = \begin{cases} 0 \quad if\ u \le -m \\ \frac{1-b}{2m(1-k)} \cdot n(v; 0, \sigma^2) \quad if\ -m < u \le -km \\ \frac{b}{2km} \cdot n(v; 0, \sigma^2) \quad if\ -km < u < km \\ \frac{1-b}{2m(1-k)} \cdot n(v; 0, \sigma^2) \quad if\ km \le u < m \\ 0 \quad if\ m \le u. \end{cases} \tag{2}$$

3.2 Recall that if $E_1$ or $E_6$ occurs, $a_1$ and $a_2$ cannot experience a simultaneous longitudinal and vertical overlap. A simultaneous overlap can occur only when $E_2$, $E_3$, $E_4$ or $E_5$ occurs; and in those cases it begins at $\max(T_b, t_b)$, and ends at $\min(T_e, t_e)$. We consider three ranges of possible values for the random variables $T_b$ and $T_e$: $[0, t_b)$, $[t_b, t_e]$, and $(t_e, \infty)$. Since $T_b < T_e$, there are six possible combinations of those ranges into which the random variables $T_b$ and $T_e$ can fall. For convenience in writing, we name the event that represents each combination:

$C_1 = \{0 \le T_b < T_e < t_b\}$; $C_2 = \{0 \le T_b < t_b \le T_e \le t_e\}$; $C_3 = \{0 \le T_b < t_b < t_e < T_e\}$;
$C_4 = \{t_b \le T_b < T_e \le t_e\}$; $C_5 = \{t_b \le T_b \le t_e < T_e\}$; $C_6 = \{t_e < T_b < T_e\}$.

The following table summarizes these six events and the corresponding intervals of simultaneous overlap.

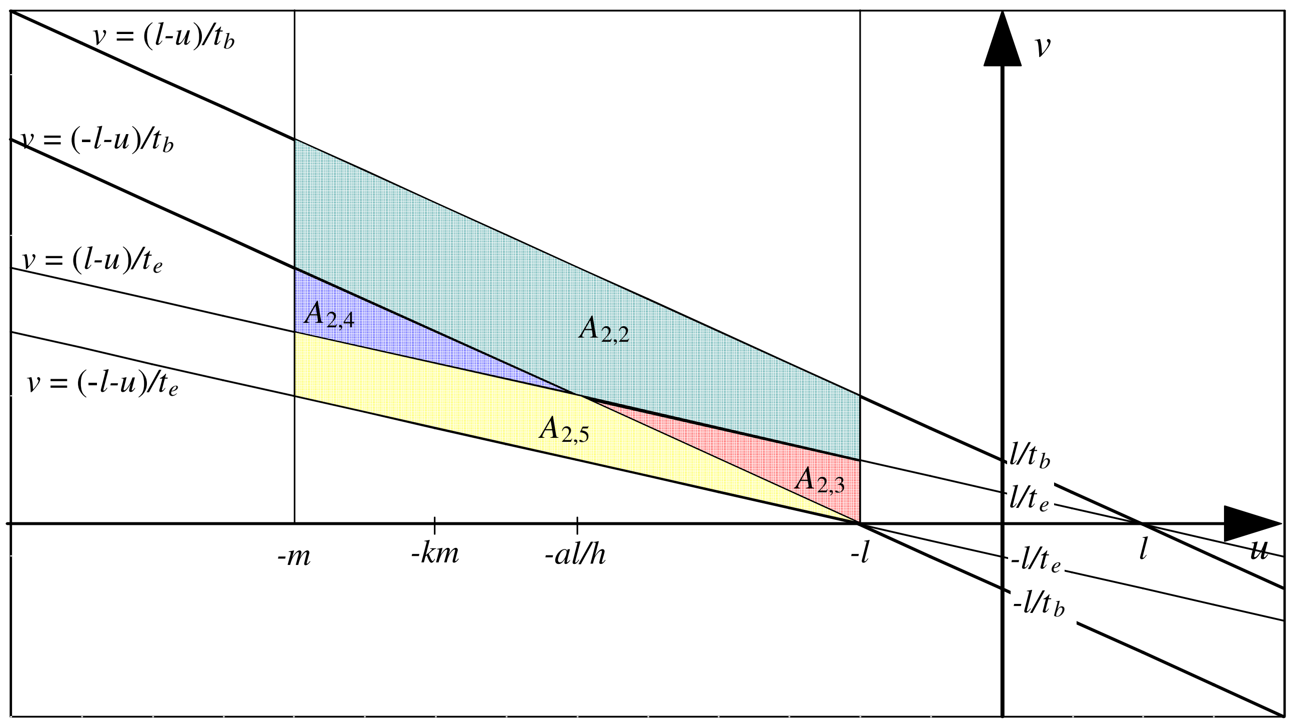
| event | $[0, t_b)$ | $[t_b, t_e]$ | $(t_e, \infty)$ | interval of simultaneous overlap $= [\max(T_b, t_b), \min(T_e, t_e)]$ |
|---|---|---|---|---|
| $C_1$ | $T_b, T_e$ | | | none |
| $C_2$ | $T_b$ | $T_e$ | | $[t_b, T_e]$ |
| $C_3$ | $T_b$ | | $T_e$ | $[t_b, t_e]$ |
| $C_4$ | | $T_b, T_e$ | | $[T_b, T_e]$ |
| $C_5$ | | $T_b$ | $T_e$ | $[T_b, t_e]$ |
| $C_6$ | | | $T_b, T_e$ | none |

We can now proceed to the task of computing the probabilities $P(E_i \cap C_j)$, for $i, j = 2, ..., 5$, i.e., for all pairs $(i, j)$ for which simultaneous longitudinal and vertical overlap may occur. However, we immediately note that since $T_b = 0$ whenever $E_3$ or $E_4$ occurs, and $t_b > 0$, the events $E_3 \cap C_4$, $E_3 \cap C_5$, $E_4 \cap C_4$ and $E_4 \cap C_5$ are all impossible, and have probability 0.

3.3 If $E_2$ occurs, then (by the definition in paragraph 2.4) $-m < U < -l$ and $V > 0$; and, as was shown in paragraph 2.7, $T_b = \dfrac{-l-U}{V}$, and $T_e = \dfrac{l-U}{V}$.

3.3.1 When $E_2 \cap C_2$ occurs, $0 < T_b = \dfrac{-l-U}{V} < t_b \le T_e = \dfrac{l-U}{V} \le t_e$. Since $V$, $t_b$ and $t_e$ are all positive, it follows that $\dfrac{-l-U}{t_b} < V \le \dfrac{l-U}{t_b}$ and $\dfrac{l-U}{t_e} \le V$. Figure 1 – which is *not* drawn to scale (since, in practice, $m$ is approximately a thousand times greater than $l$) – shows that the random vector $(U, V)$ satisfies these inequalities, and also satisfies the inequality $-m < U < -l$ (of event $E_2$) if and only if $(U, V)$ takes a value in the point set $A_{2,2} = \{(u, v)$: $-m < u < -l$ and $\max[(-l-u)/t_b, (l-u)/t_e] < v < (l-u)/t_b\}$. The lower boundary of $A_{2,2}$ is $\{(u, v)$: $-m < u < -l$ and $v = \max[(-l-u)/t_b, (l-u)/t_e]\}$. The intersection of the lines $v = (-l-u)/t_b$ and $v = (l-u)/t_e$ is the point at which $(-l-u)t_e = (l-u)t_b$, i.e., where $u = -l(t_b + t_e)/(t_e - t_b)$. Using the definitions of $t_b$

Figure 1: point sets $A_{2,2}$ , $A_{2,3}$ , $A_{2,4}$ and $A_{2,5}$



and $t_e$ in paragraph 2.10, we note that $t_b + t_e = \frac{a-h}{c} + \frac{a+h}{c} = \frac{2a}{c}$ and $t_e - t_b = \frac{a+h}{c} - \frac{a-h}{c} = \frac{2h}{c}$, so that the intersection occurs where $u = -al/h$, as shown in figure 1. The probability of the event $E_2 \cap C_2$ is the probability that the random vector $(U, V)$ takes a value in $A_{2,2}$. That is,

$$P(E_2 \cap C_2) = \iint_{A_{2,2}} f_{U,V}(u,v)\,dv\,du = \int_{-m}^{-km}\int_{(-l-u)/t_b}^{(l-u)/t_b} \frac{1-b}{2m(1-k)} n(v;0,\sigma^2)\,dv\,du$$
$$+ \int_{-km}^{-al/h}\int_{(-l-u)/t_b}^{(l-u)/t_b} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du$$
$$+ \int_{-al/h}^{-l}\int_{(l-u)/t_e}^{(l-u)/t_b} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du.$$

3.3.2 When $E_2 \cap C_3$ occurs, $0 < T_b = \frac{-l-U}{V} < t_b$ and $t_e < T_e = \frac{l-U}{V}$. Since $V$, $t_b$ and $t_e$ are all positive, it follows that $\frac{-l-U}{t_b} < V < \frac{l-U}{t_e}$. Figure 1 shows that the random vector $(U, V)$ satisfies these inequalities, and also satisfies the inequality $-m < U < -l$ (of event $E_2$) if and only if $(U, V)$ takes a value in the point set $A_{2,3} = \{(u, v)\colon -m < u < -l$ and $(-l-u)/t_b < v < (l-u)/t_e\}$.

$$P(E_2 \cap C_3) = \iint_{A_{2,3}} f_{U,V}(u,v)\,dv\,du = \int_{-al/h}^{-l}\int_{(-l-u)/t_b}^{(l-u)/t_e} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du.$$

3.3.3 When $E_2 \cap C_4$ occurs, $t_b \le \frac{-l-U}{V} < \frac{l-U}{V} \le t_e$. Since $V$, $t_b$ and $t_e$ are all positive, it follows that $\frac{l-U}{t_e} \le V \le \frac{-l-U}{t_b}$. In figure 1, the set of points $A_{2,4} = \{(u, v)\colon -m < u < -l$ and $(l-u)/t_e \le v \le (-l-u)/t_b\}$ is the set of possible values of $(U, V)$ corresponding to the event $E_2 \cap C_4$. Therefore, $P(E_2 \cap C_4) = \iint_{A_{2,4}} f_{U,V}(u,v)\,dv\,du$

$$= \int_{-m}^{-km}\int_{(l-u)/t_e}^{(-l-u)/t_b} \frac{1-b}{2m(1-k)} n(v;0,\sigma^2)\,dv\,du + \int_{-km}^{-al/h}\int_{(l-u)/t_e}^{(-l-u)/t_b} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du.$$

3.3.4 When $E_2 \cap C_5$ occurs, $t_b \le T_b = \frac{-l-U}{V} \le t_e < T_e = \frac{l-U}{V}$. Since $V$, $t_b$ and $t_e$ are all positive, it follows that $\frac{-l-U}{t_e} \le V < \frac{l-U}{t_e}$ and $V \le \frac{-l-U}{t_b}$. Thus, as shown in figure 1, the event $E_2 \cap C_5$ occurs if and only if $(U, V)$ takes a value in the point set $A_{2,5} = \{(u, v)\colon -m < u < -l$ and

$(-l-u)/t_e \le v < \min[(l-u)/t_e, (-l-u)/t_b]\}$. Therefore, $P(E_2 \cap C_5) = \iint_{A_{2,5}} f_{U,V}(u,v)\,dv\,du =$

$$\int_{-m}^{-km}\int_{(-l-u)/t_e}^{(l-u)/t_e} \frac{1-b}{2m(1-k)} n(v;0,\sigma^2)\,dv\,du$$

$$+ \int_{-km}^{-al/h}\int_{(-l-u)/t_e}^{(l-u)/t_e} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du$$

$$+ \int_{-al/h}^{-l}\int_{(-l-u)/t_e}^{(-l-u)/t_b} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du.$$

3.4 If $E_3$ occurs, then (by the definition in paragraph 2.4) $-l \le U \le l$ and $V < 0$; and, as was shown in paragraph 2.6, $T_b = 0$ and $T_e = \dfrac{-l-U}{V}$.

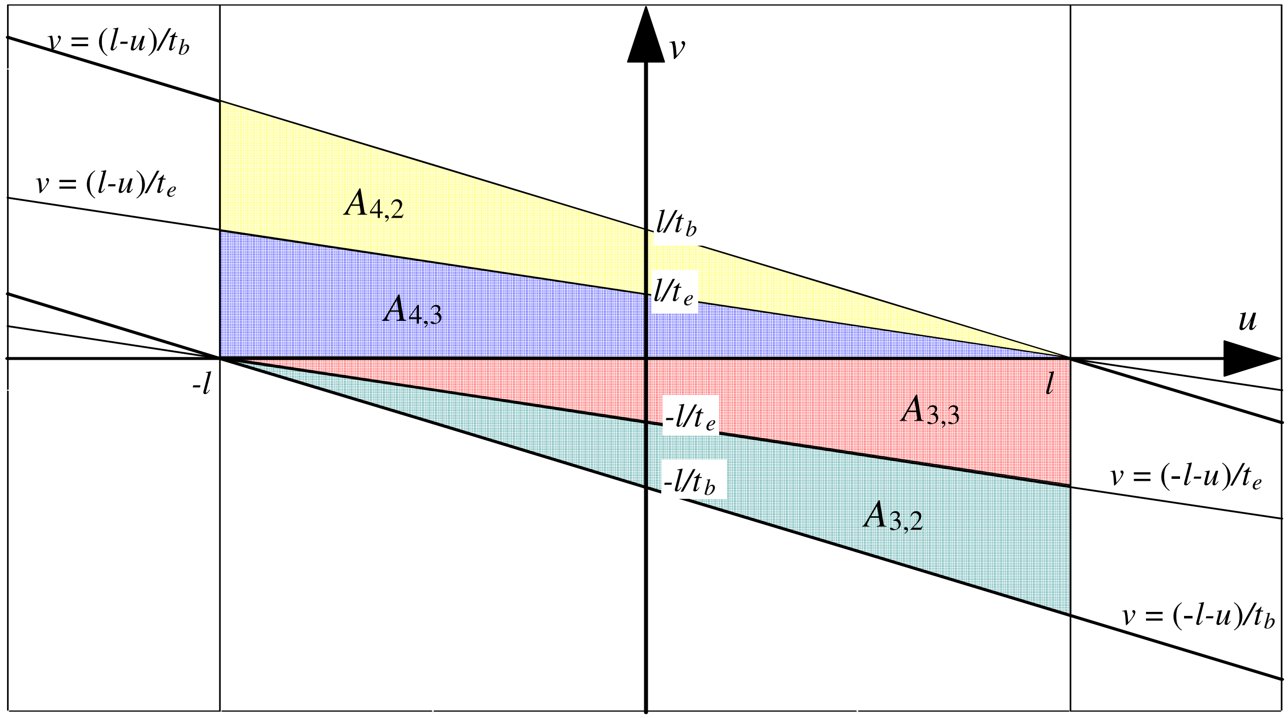
3.4.1 The event $E_3 \cap C_2$ occurs if and only if $-l \le U \le l$, $T_b = 0$, and $t_b \le T_e = \dfrac{-l-U}{V} \le t_e$. Since $V < 0$, while both $t_b$ and $t_e$ are positive, this last pair of inequalities is seen to be equivalent to $\dfrac{-l-U}{t_b} \le V \le \dfrac{-l-U}{t_e}$. The set $A_{3,2} = \{(u, v)$: $-l \le u \le l$ and $(-l-u)/t_b \le v \le (-l-u)/t_e\}$, shown in figure 2, is the set of all possible values of $(U, V)$ for which $E_3 \cap C_2$ occurs. Therefore, $P(E_3 \cap C_2) = \iint_{A_{3,2}} f_{U,V}(u,v)\,dv\,du = \int_{-l}^{l}\int_{(-l-u)/t_b}^{(-l-u)/t_e} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du$.

3.4.2 The event $E_3 \cap C_3$ occurs if and only if $-l \le U \le l$, $T_b = 0$, and $t_e < T_e = \dfrac{-l-U}{V}$. Since $V < 0$ and $t_e > 0$, this last inequality is equivalent to $\dfrac{-l-U}{t_e} < V$. Figure 2 shows point set $A_{3,3} = \{(u, v)$: $-l \le u \le l$ and $(-l-u)/t_e < v < 0\}$, i.e., the set of possible values of $(U,V)$ for which $E_3 \cap C_3$ occurs. Thus $P(E_3 \cap C_3) = \iint_{A_{3,3}} f_{U,V}(u,v)\,dv\,du = \int_{-l}^{l}\int_{(-l-u)/t_e}^{0} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du$.

3.5 If $E_4$ occurs, then (by the definition in paragraph 2.4) $-l \le U \le l$ and $V > 0$; and, as was shown in paragraph 2.6, $T_b = 0$ and $T_e = \dfrac{l-U}{V}$.

3.5.1 The event $E_4 \cap C_2$ occurs if and only if $-l \le U \le l$, $T_b = 0$, and $t_b \le T_e = \dfrac{l-U}{V} \le t_e$. Since $V$, $t_b$ and $t_e$ are all positive, this last pair of inequalities is equivalent to $\dfrac{l-U}{t_e} \le V \le \dfrac{l-U}{t_b}$. The set $A_{4,2} = \{(u, v)$: $-l \le u \le l$ and $(l-u)/t_e \le v \le (l-u)/t_b\}$, shown in figure 2, is the set of all

Figure 2: point sets $A_{3,2}$ , $A_{3,3}$ , $A_{4,2}$ and $A_{4,3}$



possible values of $(U, V)$ for which $E_4 \cap C_2$ occurs. Therefore, $P(E_4 \cap C_2) = \iint_{A_{4,2}} f_{U,V}(u,v)\,dv\,du =$
$\int_{-l}^{l}\int_{(l-u)/t_e}^{(l-u)/t_b} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du$.

3.5.2 The event $E_4 \cap C_3$ occurs if and only if $-l \le U \le l$, $T_b = 0$, and $t_e < T_e = \frac{l-U}{V}$. Since $V > 0$ and $t_e > 0$, this last inequality is equivalent to $V < \frac{l-U}{t_e}$. Figure 2 shows the point set $A_{4,3} = \{(u, v)\colon -l \le u \le l$ and $0 < v < (l-u)/t_e\}$, which is the set of values that may be assumed by the random vector $(U,V)$ in order for $E_4 \cap C_3$ to occur. Therefore, $P(E_4 \cap C_3) = \iint_{A_{4,3}} f_{U,V}(u,v)\,dv\,du = \int_{-l}^{l}\int_{0}^{(l-u)/t_e} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du$.

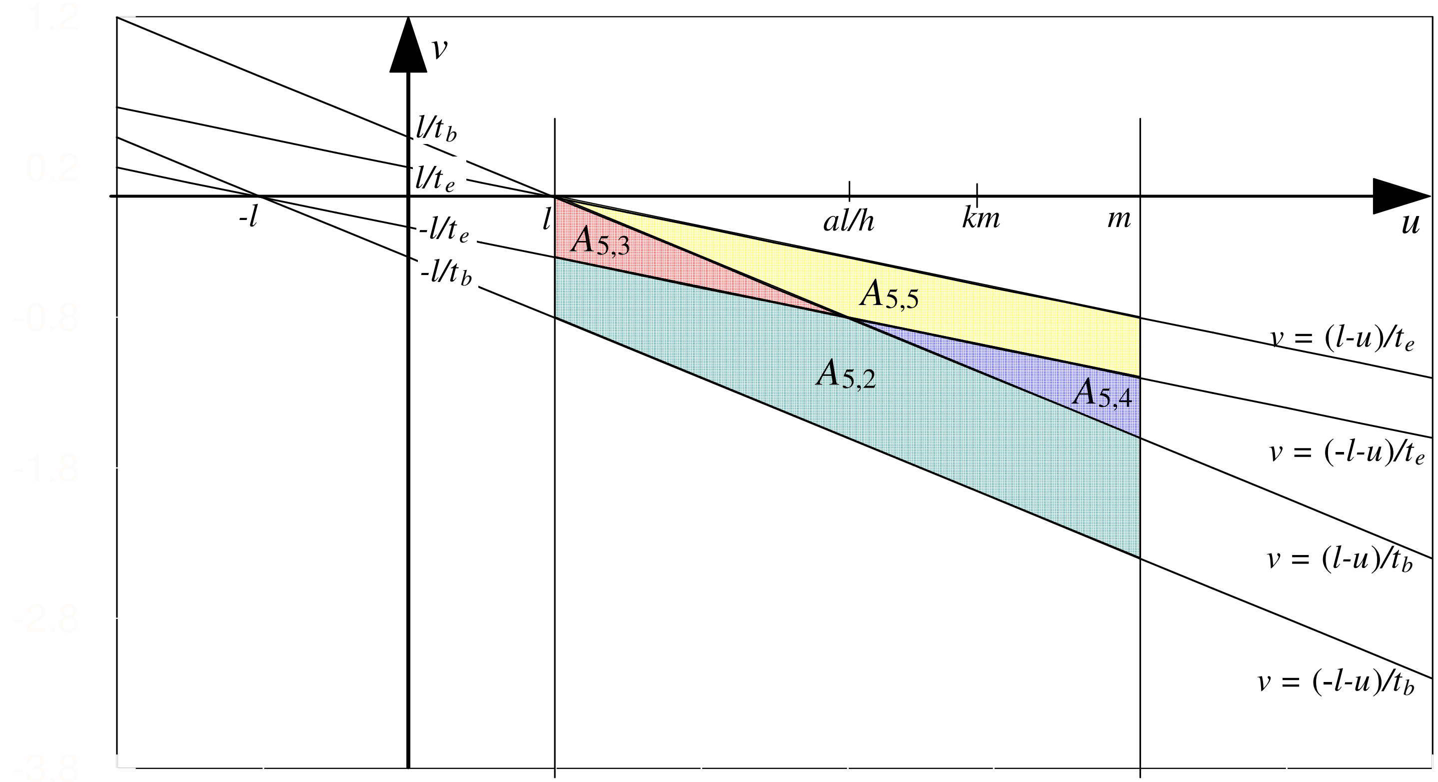
3.6 If $E_5$ occurs, then (by the definition in paragraph 2.4) $l < U < m$ and $V < 0$; and, as was shown in paragraph 2.8, $T_b = \frac{l-U}{V}$ and $T_e = \frac{-l-U}{V}$.

3.6.1 When $E_5 \cap C_2$ occurs, $0 < T_b = \frac{l-U}{V} < t_b \le T_e = \frac{-l-U}{V} \le t_e$. Since $V < 0$, but both $t_b$ and $t_e$ are positive, it follows that $\frac{-l-U}{t_b} \le V < \frac{l-U}{t_b}$ and $V \le \frac{-l-U}{t_e}$. Figure 3 shows that $(U, V)$ satisfies these inequalities, and also satisfies the inequality $l < U < m$ (of event $E_5$) if and only if it takes a value in point set $A_{5,2} = \{(u, v)\colon l < u < m$ and $(-l-u)/t_b \le v < \min[(l-u)/t_b, (-l-u)/t_e]\}$. The upper boundary of $A_{5,2}$ is $\{(u, v)\colon l < u < m$ and $v = \min[(l-u)/t_b, (-l-u)/t_e]\}$. The intersection of the lines $v = (l-u)/t_b$ and $v = (-l-u)/t_e$ is at the point where $(-l-u)t_b = (l-u)t_e$, i.e., where $u = l(t_b + t_e)/(t_e - t_b)$. As was shown in paragraph 3.3.1, $(t_b + t_e)/(t_e - t_b) = a/h$, so that the intersection occurs where $u = al/h$, as shown in figure 3. The probability of the event $E_5 \cap C_2$ is the probability that the random vector $(U, V)$ takes a value in $A_{5,2}$. That is, $P(E_5 \cap C_2) =$

$$\iint_{A_{5,2}} f_{U,V}(u,v)\,dv\,du = \int_{l}^{al/h}\int_{(-l-u)/t_b}^{(-l-u)/t_e} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du + \int_{al/h}^{km}\int_{(-l-u)/t_b}^{(l-u)/t_b} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du + \int_{km}^{m}\int_{(-l-u)/t_b}^{(l-u)/t_b} \frac{1-b}{2m(1-k)} n(v;0,\sigma^2)\,dv\,du.$$

3.6.2 When $E_5 \cap C_3$ occurs, $0 < T_b = \frac{l-U}{V} < t_b$ and $t_e < T_e = \frac{-l-U}{V}$. Since $V < 0$, but both $t_b$ and $t_e$ are positive, it follows that $\frac{-l-U}{t_e} < V < \frac{l-U}{t_b}$. Figure 3 shows that the random

Figure 3: point sets $A_{5,2}$ , $A_{5,3}$ , $A_{5,4}$ and $A_{5,5}$



vector ($U$, $V$) satisfies these inequalities, and also satisfies the inequality $l < U < m$ (of event $E_5$) if and only if ($U$, $V$) takes a value in point set $A_{5,3} = \{(u, v): l < u < m$ and $(-l-u)/t_e < v < (l-u)/t_b\}$. Therefore, $P(E_5 \cap C_3) = \iint_{A_{5,3}} f_{U,V}(u,v)\,dv\,du = \int_l^{al/h} \int_{(-l-u)/t_e}^{(l-u)/t_b} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du$.

3.6.3 When $E_5 \cap C_4$ occurs, $t_b \le T_b = \frac{l-U}{V} < T_e = \frac{-l-U}{V} \le t_e$. Since $V < 0$, but both $t_b$ and $t_e$ are positive, it follows that $\frac{l-U}{t_b} \le V \le \frac{-l-U}{t_e}$. In figure 3, the set of points $A_{5,4} = \{(u, v)$: $l < u < m$ and $(l-u)/t_b \le v \le (-l-u)/t_e\}$ is the set of possible values of ($U$, $V$) corresponding to the event $E_5 \cap C_4$. Therefore, $P(E_5 \cap C_4) = \iint_{A_{5,4}} f_{U,V}(u,v)\,dv\,du =$

$$\int_{al/h}^{km} \int_{(l-u)/t_b}^{(-l-u)/t_e} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du + \int_{km}^{m} \int_{(l-u)/t_b}^{(-l-u)/t_e} \frac{1-b}{2m(1-k)} n(v;0,\sigma^2)\,dv\,du.$$

3.6.4 When $E_5 \cap C_5$ occurs, $t_b \le T_b = \frac{l-U}{V} \le t_e < T_e = \frac{-l-U}{V}$. Since $V < 0$, but both $t_b$ and $t_e$ are positive, it follows that $\frac{-l-U}{t_e} < V \le \frac{l-U}{t_e}$ and $\frac{l-U}{t_b} \le V$. Thus, as shown in figure 3, the event $E_5 \cap C_5$ occurs if and only if ($U$, $V$) takes a value in the set $A_{5,5} = \{(u, v)$: $l < u < m$ and $\max[(-l-u)/t_e, (l-u)/t_b] < v \le (l-u)/t_e\}$. Therefore, $P(E_5 \cap C_5) = \iint_{A_{5,5}} f_{U,V}(u,v)\,dv\,du =$

$$\int_l^{al/h} \int_{(l-u)/t_b}^{(l-u)/t_e} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du$$
$$+ \int_{al/h}^{km} \int_{(-l-u)/t_e}^{(l-u)/t_e} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du$$
$$+ \int_{km}^{m} \int_{(-l-u)/t_e}^{(l-u)/t_e} \frac{1-b}{2m(1-k)} n(v;0,\sigma^2)\,dv\,du.$$

3.7 Let $O$ denote the event that $a_1$ and $a_2$ experience a simultaneous longitudinal and vertical overlap. $O$ is the union of the twelve events $E_i \cap C_j$ for which $i = 2$ or 5 and $j = 2$, 3, 4 or 5, or for which $i = 3$ or 4 and $j = 2$ or 3. From their definitions (in paragraphs 2.4 and 3.2) we see that the $E_i$ are mutually disjoint, and the $C_j$ are also mutually disjoint. Therefore, all of the events $E_i \cap C_j$ are mutually disjoint. Figures 1, 2, and 3 illustrate this property, showing that the sets $A_{i,j}$ are also mutually disjoint. Thus, we can find P($O$), the probability of simultaneous overlap, simply by adding the twelve probabilities $P(E_i \cap C_j)$. In integrating the function $n(v; 0, \sigma^2)$, we recall that if $p$ and $q$ are any two real numbers, then – by letting $x = v/\sigma$, so that $dv = \sigma\,dx$ – we have

$$\int_p^q n(v;0,\sigma^2)\,dv = \int_p^q \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{v^2}{2\sigma^2}} dv = \int_{p/\sigma}^{q/\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sigma\,dx = \int_{p/\sigma}^{q/\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{p/\sigma}^{q/\sigma} \varphi(x)\,dx$$
$$= \Phi(q/\sigma) - \Phi(p/\sigma).$$

3.7.1 We first add the probabilities that $(U, V)$ takes a value in one of the sets $A_{2,\,j}$, which are illustrated in figure 1. Since those sets are disjoint, $\sum_{j=2}^{5} \mathrm{P}(E_2 \cap C_j) = \sum_{j=2}^{5} \iint_{A_{2,j}} f_{U,V}(u,v)\,dv\,du$

$$= \iint_{A_{2,2} \cup A_{2,3} \cup A_{2,4} \cup A_{2,5}} f_{U,V}(u,v)\,dv\,du$$

$$= \int_{-m}^{-km} \int_{(-l-u)/t_e}^{(l-u)/t_b} f_{U,V}(u,v)\,dv\,du \;+\; \int_{-km}^{-l} \int_{(-l-u)/t_e}^{(l-u)/t_b} f_{U,V}(u,v)\,dv\,du$$

$$= \int_{-m}^{-km} \int_{(-l-u)/t_e}^{(l-u)/t_b} \frac{1-b}{2m(1-k)}\, n(v;0,\sigma^2)\,dv\,du \;+\; \int_{-km}^{-l} \int_{(-l-u)/t_e}^{(l-u)/t_b} \frac{b}{2mk}\, n(v;0,\sigma^2)\,dv\,du$$

$$= \frac{1-b}{2m(1-k)} \int_{-m}^{-km} \left[ \Phi\left( \frac{l-u}{\sigma t_b} \right) - \Phi\left( \frac{-l-u}{\sigma t_e} \right) \right] du \;+\; \frac{b}{2km} \int_{-km}^{-l} \left[ \Phi\left( \frac{l-u}{\sigma t_b} \right) - \Phi\left( \frac{-l-u}{\sigma t_e} \right) \right] du$$

$$= \frac{1-b}{2m(1-k)} \left[ \int_{-m}^{-km} \Phi\left( \frac{l-u}{\sigma t_b} \right) du - \int_{-m}^{-km} \Phi\left( \frac{-l-u}{\sigma t_e} \right) du \right]$$
$$+ \; \frac{b}{2km} \left[ \int_{-km}^{-l} \Phi\left( \frac{l-u}{\sigma t_b} \right) du - \int_{-km}^{-l} \Phi\left( \frac{-l-u}{\sigma t_e} \right) du \right].$$

3.7.2 We next add the probabilities that $(U, V)$ takes a value in one of the sets $A_{3,\,j}$ or $A_{4,\,j}$, illustrated in figure 2. Since those sets are also disjoint, $\sum_{\substack{i=3,4 \\ j=2,3}} \mathrm{P}(E_i \cap C_j) = \sum_{\substack{i=3,4 \\ j=2,3}} \iint_{A_{i,j}} f_{U,V}(u,v)\,dv\,du$

$$= \iint_{A_{3,2} \cup A_{3,3} \cup A_{4,2} \cup A_{4,3}} f_{U,V}(u,v)\,dv\,du \;=\; \int_{-l}^{l} \int_{(-l-u)/t_b}^{(l-u)/t_b} \frac{b}{2km}\, n(v;0,\sigma^2)\,dv\,du$$

$$= \frac{b}{2km} \int_{-l}^{l} \left[ \Phi\left( \frac{l-u}{\sigma t_b} \right) - \Phi\left( \frac{-l-u}{\sigma t_b} \right) \right] du$$

$$= \frac{b}{2km} \left[ \int_{-l}^{l} \Phi\left( \frac{l-u}{\sigma t_b} \right) du - \int_{-l}^{l} \Phi\left( \frac{-l-u}{\sigma t_b} \right) du \right].$$

3.7.3 Finally we add the probabilities that $(U, V)$ takes a value in one of the sets $A_{5,j}$, which are illustrated in figure 3. Again the sets are disjoint; so $\sum_{j=2}^{5} \mathrm{P}(E_5 \cap C_j) = \sum_{j=2}^{5} \iint_{A_{5,j}} f_{U,V}(u,v)\,dv\,du$

$$= \iint_{A_{5,2} \cup A_{5,3} \cup A_{5,4} \cup A_{5,5}} f_{U,V}(u,v)\,dv\,du$$

$$= \int_{l}^{km} \int_{(-l-u)/t_b}^{(l-u)/t_e} f_{U,V}(u,v)\,dv\,du \; + \int_{km}^{m} \int_{(-l-u)/t_b}^{(l-u)/t_e} f_{U,V}(u,v)\,dv\,du$$

$$= \int_{l}^{km} \int_{(-l-u)/t_b}^{(l-u)/t_e} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du \; + \int_{km}^{m} \int_{(-l-u)/t_b}^{(l-u)/t_e} \frac{1-b}{2m(1-k)}\, n(v;0,\sigma^2)\,dv\,du$$

$$= \frac{b}{2km}\int_{l}^{km}\left[\Phi\left(\frac{l-u}{\sigma t_e}\right) - \Phi\left(\frac{-l-u}{\sigma t_b}\right)\right]du \; + \frac{1-b}{2m(1-k)}\int_{km}^{m}\left[\Phi\left(\frac{l-u}{\sigma t_e}\right) - \Phi\left(\frac{-l-u}{\sigma t_b}\right)\right]du$$

$$= \frac{b}{2km}\left[\int_{l}^{km}\Phi\left(\frac{l-u}{\sigma t_e}\right)du - \int_{l}^{km}\Phi\left(\frac{-l-u}{\sigma t_b}\right)du\right]$$
$$+ \frac{1-b}{2m(1-k)}\left[\int_{km}^{m}\Phi\left(\frac{l-u}{\sigma t_e}\right)du - \int_{km}^{m}\Phi\left(\frac{-l-u}{\sigma t_b}\right)du\right].$$

3.7.4 Combining results from the last three paragraphs gives us P($O$) =

$$\frac{1-b}{2m(1-k)}\left[\int_{-m}^{-km}\Phi\left(\frac{l-u}{\sigma t_b}\right)du - \int_{-m}^{-km}\Phi\left(\frac{-l-u}{\sigma t_e}\right)du\right] + \frac{b}{2km}\left[\int_{-km}^{-l}\Phi\left(\frac{l-u}{\sigma t_b}\right)du - \int_{-km}^{-l}\Phi\left(\frac{-l-u}{\sigma t_e}\right)du\right]$$

$$+ \frac{b}{2km}\left[\int_{-l}^{l}\Phi\left(\frac{l-u}{\sigma t_b}\right)du - \int_{-l}^{l}\Phi\left(\frac{-l-u}{\sigma t_b}\right)du\right]$$

$$+ \frac{b}{2km}\left[\int_{l}^{km}\Phi\left(\frac{l-u}{\sigma t_e}\right)du - \int_{l}^{km}\Phi\left(\frac{-l-u}{\sigma t_b}\right)du\right] + \frac{1-b}{2m(1-k)}\left[\int_{km}^{m}\Phi\left(\frac{l-u}{\sigma t_e}\right)du - \int_{km}^{m}\Phi\left(\frac{-l-u}{\sigma t_b}\right)du\right]$$

$$= \frac{1-b}{2m(1-k)}\left[\int_{-m}^{-km}\Phi\left(\frac{l-u}{\sigma t_b}\right)du - \int_{-m}^{-km}\Phi\left(\frac{-l-u}{\sigma t_e}\right)du + \int_{km}^{m}\Phi\left(\frac{l-u}{\sigma t_e}\right)du - \int_{km}^{m}\Phi\left(\frac{-l-u}{\sigma t_b}\right)du\right]$$
$$+ \frac{b}{2km}\left[\int_{-km}^{-l}\Phi\left(\frac{l-u}{\sigma t_b}\right)du - \int_{-km}^{-l}\Phi\left(\frac{-l-u}{\sigma t_e}\right)du + \int_{-l}^{l}\Phi\left(\frac{l-u}{\sigma t_b}\right)du\right.$$
$$\left. - \int_{-l}^{l}\Phi\left(\frac{-l-u}{\sigma t_b}\right)du + \int_{l}^{km}\Phi\left(\frac{l-u}{\sigma t_e}\right)du - \int_{l}^{km}\Phi\left(\frac{-l-u}{\sigma t_b}\right)du\right].$$

$$= \frac{1-b}{2m(1-k)}\left[\int_{-m}^{-km}\Phi\left(\frac{l-u}{\sigma t_b}\right)du - \int_{-m}^{-km}\Phi\left(\frac{-l-u}{\sigma t_e}\right)du + \int_{km}^{m}\Phi\left(\frac{l-u}{\sigma t_e}\right)du - \int_{km}^{m}\Phi\left(\frac{-l-u}{\sigma t_b}\right)du\right]$$

$$+ \frac{b}{2km}\left[\int_{-km}^{l}\Phi\left(\frac{l-u}{\sigma t_b}\right)du - \int_{-km}^{-l}\Phi\left(\frac{-l-u}{\sigma t_e}\right)du + \int_{l}^{km}\Phi\left(\frac{l-u}{\sigma t_e}\right)du - \int_{-l}^{km}\Phi\left(\frac{-l-u}{\sigma t_b}\right)du\right].$$

In order to simplify this last expression we make appropriate substitutions for the arguments of the standard normal distribution function Φ. In the two integrals for which $\frac{l-u}{\sigma t_b}$ is the argument of Φ, we let $w = \frac{l-u}{\sigma t_b}$. Then $du = -\sigma t_b dw$; when $u = -m$, $w = \frac{m+l}{\sigma t_b}$; when $u = -km$, $w = \frac{km+l}{\sigma t_b}$; and when $u = l$, $w = 0$. In the two integrals for which $\frac{-l-u}{\sigma t_e}$ is the argument of Φ, we let $x = \frac{-l-u}{\sigma t_e}$. Then $du = -\sigma t_e dx$; when $u = -m$, $x = \frac{m-l}{\sigma t_e}$; when $u = -km$, $x = \frac{km-l}{\sigma t_e}$; and when $u = -l$, $x = 0$. In the two integrals for which $\frac{l-u}{\sigma t_e}$ is the argument of Φ, we let $y = \frac{l-u}{\sigma t_e} = -\frac{u-l}{\sigma t_e}$. Then $du = -\sigma t_e dy$; when $u = l$, $y = 0$; when $u = km$, $y = -\frac{km-l}{\sigma t_e}$; and when $u = m$, $y = -\frac{m-l}{\sigma t_e}$. In the two integrals for which $\frac{-l-u}{\sigma t_b}$ is the argument of Φ, we let $z = \frac{-l-u}{\sigma t_b} = -\frac{u+l}{\sigma t_b}$. Then $du = -\sigma t_b dz$; when $u = -l$, $z = 0$; when $u = km$, $z = -\frac{km+l}{\sigma t_b}$; and when $u = m$, $z = -\frac{m+l}{\sigma t_b}$. Applying these substitutions we find that P($O$) =

$$\frac{1-b}{2m(1-k)}\left[\int_{\frac{m+l}{\sigma t_b}}^{\frac{km+l}{\sigma t_b}}\Phi(w)(-\sigma t_b dw) - \int_{\frac{m-l}{\sigma t_e}}^{\frac{km-l}{\sigma t_e}}\Phi(x)(-\sigma t_e dx) + \int_{-\frac{km-l}{\sigma t_e}}^{-\frac{m-l}{\sigma t_e}}\Phi(y)(-\sigma t_e dy) - \int_{-\frac{km+l}{\sigma t_b}}^{-\frac{m+l}{\sigma t_b}}\Phi(z)(-\sigma t_b dz)\right]$$

$$+ \frac{b}{2km}\left[\int_{\frac{km+l}{\sigma t_b}}^{0}\Phi(w)(-\sigma t_b dw) - \int_{\frac{km-l}{\sigma t_e}}^{0}\Phi(x)(-\sigma t_e dx) + \int_{0}^{-\frac{km-l}{\sigma t_e}}\Phi(y)(-\sigma t_e dy) - \int_{0}^{-\frac{km+l}{\sigma t_b}}\Phi(z)(-\sigma t_b dz)\right]$$

$$= \frac{1-b}{2m(1-k)}\left[\sigma t_b \int_{\frac{km+l}{\sigma t_b}}^{\frac{m+l}{\sigma t_b}} \Phi(w)dw - \sigma t_b \int_{-\frac{m+l}{\sigma t_b}}^{-\frac{km+l}{\sigma t_b}} \Phi(z)dz + \sigma t_e \int_{-\frac{m-l}{\sigma t_e}}^{-\frac{km-l}{\sigma t_e}} \Phi(y)dy - \sigma t_e \int_{\frac{km-l}{\sigma t_e}}^{\frac{m-l}{\sigma t_e}} \Phi(x)dx\right]$$

$$+ \frac{b}{2km}\left[\sigma t_b \int_0^{\frac{km+l}{\sigma t_b}} \Phi(w)dw - \sigma t_b \int_{-\frac{km+l}{\sigma t_b}}^{0} \Phi(z)dz + \sigma t_e \int_{-\frac{km-l}{\sigma t_e}}^{0} \Phi(y)dy - \sigma t_e \int_0^{\frac{km-l}{\sigma t_e}} \Phi(x)dx\right]$$

$$= \frac{1-b}{2m(1-k)}\left[\sigma t_b\left(\Psi\left(\frac{m+l}{\sigma t_b}\right) - \Psi\left(\frac{km+l}{\sigma t_b}\right)\right) - \sigma t_b\left(\Psi\left(-\frac{km+l}{\sigma t_b}\right) - \Psi\left(-\frac{m+l}{\sigma t_b}\right)\right)\right.$$

$$\left. + \sigma t_e\left(\Psi\left(-\frac{km-l}{\sigma t_e}\right) - \Psi\left(-\frac{m-l}{\sigma t_e}\right)\right) - \sigma t_e\left(\Psi\left(\frac{m-l}{\sigma t_e}\right) - \Psi\left(\frac{km-l}{\sigma t_e}\right)\right)\right]$$

$$+ \frac{b}{2km}\left[\sigma t_b\left(\Psi\left(\frac{km+l}{\sigma t_b}\right) - \Psi(0)\right) - \sigma t_b\left(\Psi(0) - \Psi\left(-\frac{km+l}{\sigma t_b}\right)\right)\right.$$

$$\left. + \sigma t_e\left(\Psi(0) - \Psi\left(-\frac{km-l}{\sigma t_e}\right)\right) - \sigma t_e\left(\Psi\left(\frac{km-l}{\sigma t_e}\right) - \Psi(0)\right)\right]$$

$$= \frac{1-b}{2m(1-k)}\left\{\sigma t_b\left[\left(\Psi\left(\frac{m+l}{\sigma t_b}\right) + \Psi\left(-\frac{m+l}{\sigma t_b}\right)\right) - \left(\Psi\left(\frac{km+l}{\sigma t_b}\right) + \Psi\left(-\frac{km+l}{\sigma t_b}\right)\right)\right]\right.$$

$$\left. + \sigma t_e\left[\left(\Psi\left(\frac{km-l}{\sigma t_e}\right) + \Psi\left(-\frac{km-l}{\sigma t_e}\right)\right) - \left(\Psi\left(\frac{m-l}{\sigma t_e}\right) + \Psi\left(-\frac{m-l}{\sigma t_e}\right)\right)\right]\right\}$$

$$+ \frac{b}{2km}\left\{\sigma t_b\left[\left(\Psi\left(\frac{km+l}{\sigma t_b}\right) - \Psi(0)\right) + \left(\Psi\left(-\frac{km+l}{\sigma t_b}\right) - \Psi(0)\right)\right]\right.$$

$$\left. - \sigma t_e\left[\left(\Psi\left(\frac{km-l}{\sigma t_e}\right) - \Psi(0)\right) + \left(\Psi\left(-\frac{km-l}{\sigma t_e}\right) - \Psi(0)\right)\right]\right\} \qquad (3a)$$

3.7.5 Taking advantage of the symmetry of the normal density function, we note that for any real $x$, $\Psi(0) - \Psi(-x) = \int_{-x}^{0}\Phi(t)\,dt = \int_{-x}^{0}[1-\Phi(-t)]\,dt$ = (for $s = -t$ ) $\int_{x}^{0}[1-\Phi(s)](-ds) = \int_{0}^{x}[1-\Phi(s)]\,ds = [s]_0^x - [\Psi(s)]_0^x = x - [\Psi(x) - \Psi(0)]$. From this result we draw three conclusions:

$$[\Psi(x) - \Psi(0)] + [\Psi(-x) - \Psi(0)] = [\Psi(x) - \Psi(0)] - [\Psi(0) - \Psi(-x)]$$
$$= [\Psi(x) - \Psi(0)] - \{x - [\Psi(x) - \Psi(0)]\} = 2\cdot[\Psi(x) - \Psi(0)] - x\,; \tag{4a}$$

$$\Psi(0) - \Psi(-x) = x - \Psi(x) + \Psi(0), \text{ from which } \Psi(x) - \Psi(-x) = x\,; \text{ and} \tag{4b}$$

(since equation (4b) tells us that $\Psi(-x) = \Psi(x) - x$ )
$$\Psi(x) + \Psi(-x) = 2\Psi(x) - x. \tag{4c}$$

3.7.6 Applying formula (4c) to the first two lines of the right side of equation (3a), and applying formula (4a) to its last two lines, we rewrite that equation as P($O$) =

$$\frac{1-b}{2m(1-k)}\left\{\sigma t_b\left[\left(2\Psi\left(\frac{m+l}{\sigma t_b}\right) - \frac{m+l}{\sigma t_b}\right) - \left(2\Psi\left(\frac{km+l}{\sigma t_b}\right) - \frac{km+l}{\sigma t_b}\right)\right]\right.$$
$$\left. + \;\sigma t_e\left[\left(2\Psi\left(\frac{km-l}{\sigma t_e}\right) - \frac{km-l}{\sigma t_e}\right) - \left(2\Psi\left(\frac{m-l}{\sigma t_e}\right) - \frac{m-l}{\sigma t_e}\right)\right]\right\}$$
$$+ \;\frac{b}{2km}\left\{\sigma t_b\left[2\cdot\left(\Psi\left(\frac{km+l}{\sigma t_b}\right) - \Psi(0)\right) - \frac{km+l}{\sigma t_b}\right] - \sigma t_e\left[2\cdot\left(\Psi\left(\frac{km-l}{\sigma t_e}\right) - \Psi(0)\right) - \frac{km-l}{\sigma t_e}\right]\right\}$$

$$= \;\frac{1-b}{2m(1-k)}\left\{2\sigma t_b\Psi\left(\frac{m+l}{\sigma t_b}\right) - m - l - 2\sigma t_b\Psi\left(\frac{km+l}{\sigma t_b}\right) + km + l\right.$$
$$\left. + \;2\sigma t_e\Psi\left(\frac{km-l}{\sigma t_e}\right) - km + l - 2\sigma t_e\Psi\left(\frac{m-l}{\sigma t_e}\right) + m - l\right\}$$
$$+ \;\frac{b}{2km}\left\{2\sigma t_b\Psi\left(\frac{km+l}{\sigma t_b}\right) - 2\sigma t_b\Psi(0) - km - l - 2\sigma t_e\Psi\left(\frac{km-l}{\sigma t_e}\right) + 2\sigma t_e\Psi(0) + km - l\right\}$$

$$= \;\frac{1-b}{2m(1-k)}\left\{2\sigma t_b\Psi\left(\frac{m+l}{\sigma t_b}\right) - 2\sigma t_b\Psi\left(\frac{km+l}{\sigma t_b}\right) + 2\sigma t_e\Psi\left(\frac{km-l}{\sigma t_e}\right) - 2\sigma t_e\Psi\left(\frac{m-l}{\sigma t_e}\right)\right\}$$
$$+ \;\frac{b}{2km}\left\{2\sigma t_b\Psi\left(\frac{km+l}{\sigma t_b}\right) - 2\sigma t_e\Psi\left(\frac{km-l}{\sigma t_e}\right) + 2\sigma(t_e - t_b)\Psi(0) - 2l\right\}$$

$$= \sigma t_b \Psi\left(\frac{km+l}{\sigma t_b}\right)\cdot\left[\frac{b}{km} - \frac{1-b}{m(1-k)}\right] - \sigma t_e \Psi\left(\frac{km-l}{\sigma t_e}\right)\cdot\left[\frac{b}{km} - \frac{1-b}{m(1-k)}\right]$$

$$+ \frac{1-b}{m(1-k)}\left[\sigma t_b \Psi\left(\frac{m+l}{\sigma t_b}\right) - \sigma t_e \Psi\left(\frac{m-l}{\sigma t_e}\right)\right] + \frac{b}{mk}\left[\sigma(t_e - t_b)\Psi(0) - l\right]$$

$$= \left[\sigma t_b \Psi\left(\frac{km+l}{\sigma t_b}\right) - \sigma t_e \Psi\left(\frac{km-l}{\sigma t_e}\right)\right]\cdot\left[\frac{b(1-k)-(1-b)k}{mk(1-k)}\right]$$

$$+ \frac{1-b}{m(1-k)}\left[\sigma t_b \Psi\left(\frac{m+l}{\sigma t_b}\right) - \sigma t_e \Psi\left(\frac{m-l}{\sigma t_e}\right)\right] + \frac{b}{mk}\left[\sigma(t_e - t_b)\Psi(0) - l\right]$$

$$= \frac{b-k}{mk(1-k)}\left[\sigma t_b \Psi\left(\frac{km+l}{\sigma t_b}\right) - \sigma t_e \Psi\left(\frac{km-l}{\sigma t_e}\right)\right]$$

$$+ \frac{1-b}{m(1-k)}\left[\sigma t_b \Psi\left(\frac{m+l}{\sigma t_b}\right) - \sigma t_e \Psi\left(\frac{m-l}{\sigma t_e}\right)\right] + \frac{b}{mk}\left[\sigma(t_e - t_b)\Psi(0) - l\right]. \qquad (3b)$$

3.8 Since we can't write $\Phi(t)$ in closed form, we also can't obtain a closed-form expression for $\Psi(x) = \int_{-\infty}^{x} \Phi(t)\,dt$. However, we can obtain values of $\Psi(x)$ by using the Taylor series expansion $\Psi(x) = \frac{1}{\sqrt{2\pi}} + \frac{x}{2} + \frac{1}{\sqrt{2\pi}}\left[\frac{x^2}{2} + \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k+2}}{(2k+1)(2k+2)\prod_{j=1}^{k}(2j)}\right]$. In particular, this formula shows that $\Psi(0) = 1/\sqrt{2\pi}$. Thus we can simplify equation (3b) to obtain P($O$) =

$$= \frac{b-k}{mk(1-k)}\left[\sigma t_b \Psi\left(\frac{km+l}{\sigma t_b}\right) - \sigma t_e \Psi\left(\frac{km-l}{\sigma t_e}\right)\right]$$

$$+ \frac{1-b}{m(1-k)}\left[\sigma t_b \Psi\left(\frac{m+l}{\sigma t_b}\right) - \sigma t_e \Psi\left(\frac{m-l}{\sigma t_e}\right)\right] + \frac{b}{mk}\left[\frac{\sigma(t_e - t_b)}{\sqrt{2\pi}} - l\right]. \qquad (3c)$$

3.9 The Taylor series expansion for $\Psi(x)$ consists largely of a sum of alternating positive and negative terms; and when $|x| > 6$, the absolute values of some terms become large enough for a spreadsheet implementation to lose precision. Fortunately, we can apply two simple approximations. The following table gives values of $\Psi(x)$ for integer values of $x$ ranging from $-6$ to 6:

| $x$ | $\Psi(x)$ | $x$ | $\Psi(x)$ |
|---|---|---|---|
| -6 | 1.466569088393E-10 | 1 | 1.083315470588E+00 |
| -5 | 5.346160403263E-08 | 2 | 2.008490702617E+00 |
| -4 | 7.145258432928E-06 | 3 | 3.000382154317E+00 |
| -3 | 3.821543170477E-04 | 4 | 4.000007145258E+00 |
| -2 | 8.490702616830E-03 | 5 | 5.000000053462E+00 |
| -1 | 8.331547058769E-02 | 6 | 6.000000000147E+00 |
| 0 | 3.989422804014E-01 | | |

The table suggests that as $x$ becomes more and more negatively large, $\Psi(x)$ rapidly approaches 0 (from above); and as $x$ becomes more and more positively large, $\Psi(x)$ rapidly approaches $x$ (from above). We intuitively expect $\Psi(x)$ to approach 0 as $x$ approaches $-\infty$, since $\Phi$ is a continuous distribution function. Figure 4 can give us an intuitive understanding of the behavior of $\Psi$ as $x$ becomes positively large. The figure shows $\Phi$, the standard normal distribution function, as a blue curve. $\Psi(x)$ is the area under the curve, from $-\infty$ to $x$. $\Psi(x)$ is illustrated as the sum of the areas of regions $R_1$ and $R_2$. Since $1-\Phi(x) = \Phi(-x)$ for all real $x$, region $R_3$ looks more and more like (an inverted copy of) region $R_1$ as $x$ becomes (positively) large. Therefore, as $x$ becomes large, the sum of the areas of $R_1$ and $R_2$ approaches (from above) the sum of the areas of $R_2$ and $R_3$. The sum of the areas of $R_2$ and $R_3$ is simply $\int_0^x \Phi(t)dt + \int_0^x \left(1-\Phi(t)\right)dt = \int_0^x \left(\Phi(t)+1-\Phi(t)\right)dt = \int_0^x dt = x.$ In other words, $\Psi(x)$ approaches $x$ (from above); and, as is shown in the table, the approach is quite rapid. (Assuming that $\Psi(x)$ approaches 0 as $x$ approaches $-\infty$, and invoking equation (4b), gives us another way to see that $\Psi(x)$ approaches $x$ as $x$ becomes positively large.)

3.10 Figure 5 shows a spreadsheet used to compute P($O$). In this particular example $(m+l)/(\sigma t_b)$ and $(m-l)/(\sigma t_e)$ are far greater than 6; and so we use the approximation $\Psi(x) \approx x$ in order to compute $\Psi[(m+l)/(\sigma t_b)]$ and $\Psi[(m-l)/(\sigma t_e)]$. However, $(km+l)/(\sigma t_b)$ and $(km-l)/(\sigma t_e)$ are less than 6; so we compute the values of $\Psi$ at those arguments by applying the Taylor series. The numbers shown on the spreadsheet indicate that even for an argument as small as 4.82, the approximation gives the same values of the first five decimal places, and would give the same value if rounded to the sixth decimal place. It is important to recognize that, while the parameter values used in figure 5 are not unrealistic, they were simply chosen to be illustrative, and are not based on an empirical study. In particular, the value of 35 kts used for $\sigma$, the standard deviation of the difference between the airplanes' speeds, is approximately 0.06 mach at the flight levels where modern transport airplanes typically cruise.

3.11 If all arguments of $\Psi$ in equation (3c) are greater than 6, we can safely write P($O$) $\approx$

$$\frac{b-k}{mk(1-k)}\left[\sigma t_b\cdot\frac{km+l}{\sigma t_b} - \sigma t_e\cdot\frac{km-l}{\sigma t_e}\right] + \frac{1-b}{m(1-k)}\left[\sigma t_b\cdot\frac{m+l}{\sigma t_b} - \sigma t_e\cdot\frac{m-l}{\sigma t_e}\right] + \frac{b}{mk}\left[\frac{\sigma(t_e-t_b)}{\sqrt{2\pi}} - l\right]$$

$$= \frac{b-k}{mk(1-k)}[(km+l)-(km-l)] + \frac{1-b}{m(1-k)}[(m+l)-(m-l)] + \frac{b}{mk}\cdot\frac{\sigma(t_e-t_b)}{\sqrt{2\pi}} - \frac{b}{mk}\cdot l$$

$$= \frac{b-k}{mk(1-k)}\cdot 2l + \frac{k-bk}{mk(1-k)}\cdot 2l + \frac{b}{mk}\cdot\frac{\sigma(t_e-t_b)}{\sqrt{2\pi}} - \frac{b-bk}{mk(1-k)}\cdot l$$

$$= \frac{2b-2k+2k-2bk-b+bk}{mk(1-k)}\cdot l + \frac{b\sigma(t_e-t_b)}{mk\sqrt{2\pi}} = \frac{b-bk}{mk(1-k)}\cdot l + \frac{b\sigma(t_e-t_b)}{mk\sqrt{2\pi}}$$

$$= \frac{bl}{mk} + \frac{b\sigma(t_e-t_b)}{mk\sqrt{2\pi}} = \frac{b}{mk}\left(l + \frac{\sigma(t_e-t_b)}{\sqrt{2\pi}}\right). \tag{3d}$$

Since (as was shown in paragraph 3.3.1) $t_e - t_b = 2h/c$, we can rewrite approximation (3d) as

$$\mathrm{P}(O) \approx \frac{b}{mk}\left(l + \frac{2\sigma h}{\sqrt{2\pi}\,c}\right); \tag{3e}$$

and since (as was shown in paragraph 3.3.1) we also have $t_b + t_e = 2a/c$ – from which it immediately follows that $2/c = (t_b + t_e)/a$ – we can also rewrite the approximation as

$$\mathrm{P}(O) \approx \frac{b}{mk}\left(l + \frac{(t_b+t_e)\sigma h}{\sqrt{2\pi}\,a}\right). \tag{3f}$$

3.12 Figure 6 shows curves generated by repeated applications of a spreadsheet similar to the one shown in figure 5. The curves indicate that increases in vertical speed, $c$, lead to decreases in the probability of simultaneous longitudinal and vertical overlap. This consequence is mathematically clear from approximation (3e); and it is intuitively reasonable, since: (1) the probability of simultaneous overlap varies in the same sense as the duration of the time period in which the airplanes are in vertical overlap; and (2) for constant aircraft height $h$ and varying vertical speed $c$, the time spent in vertical overlap, $2h/c$ $(= t_e - t_b)$, is simply one branch of a hyperbola. Thus the shape of the curves in figure 6 is not surprising. What is especially noteworthy is that all six curves become virtually coincident once the climb or descent speed, $c$, exceeds a few hundred feet per minute. At all vertical speeds except the smallest ones shown in the figure, the difference, $a$, between the initial altitudes of the airplanes, has a negligibly small effect on the probability of simultaneous overlap.

3.13 Figure 6 also suggests a means of minimizing risk. If it is possible to choose "worst-case" values of the parameters shown at the top of the spreadsheet in figure 5, and if such values produce a graph similar to figure 6, then it should also be possible to choose a minimum acceptable speed of climb or descent, i.e., a vertical speed that will reduce P($O$), the probability of simultaneous longitudinal and vertical overlap, to a value that is close to its minimum. For example, in figure 6 we might set the minimum acceptable vertical speed (in feet per minute) to one-fifth of the initial vertical separation (in feet). (Thus the airplanes would enter into vertical overlap no more than 5 minutes after the start of the climb or descent.) This minimum vertical speed could then be incorporated into the rules for executing an ITP.

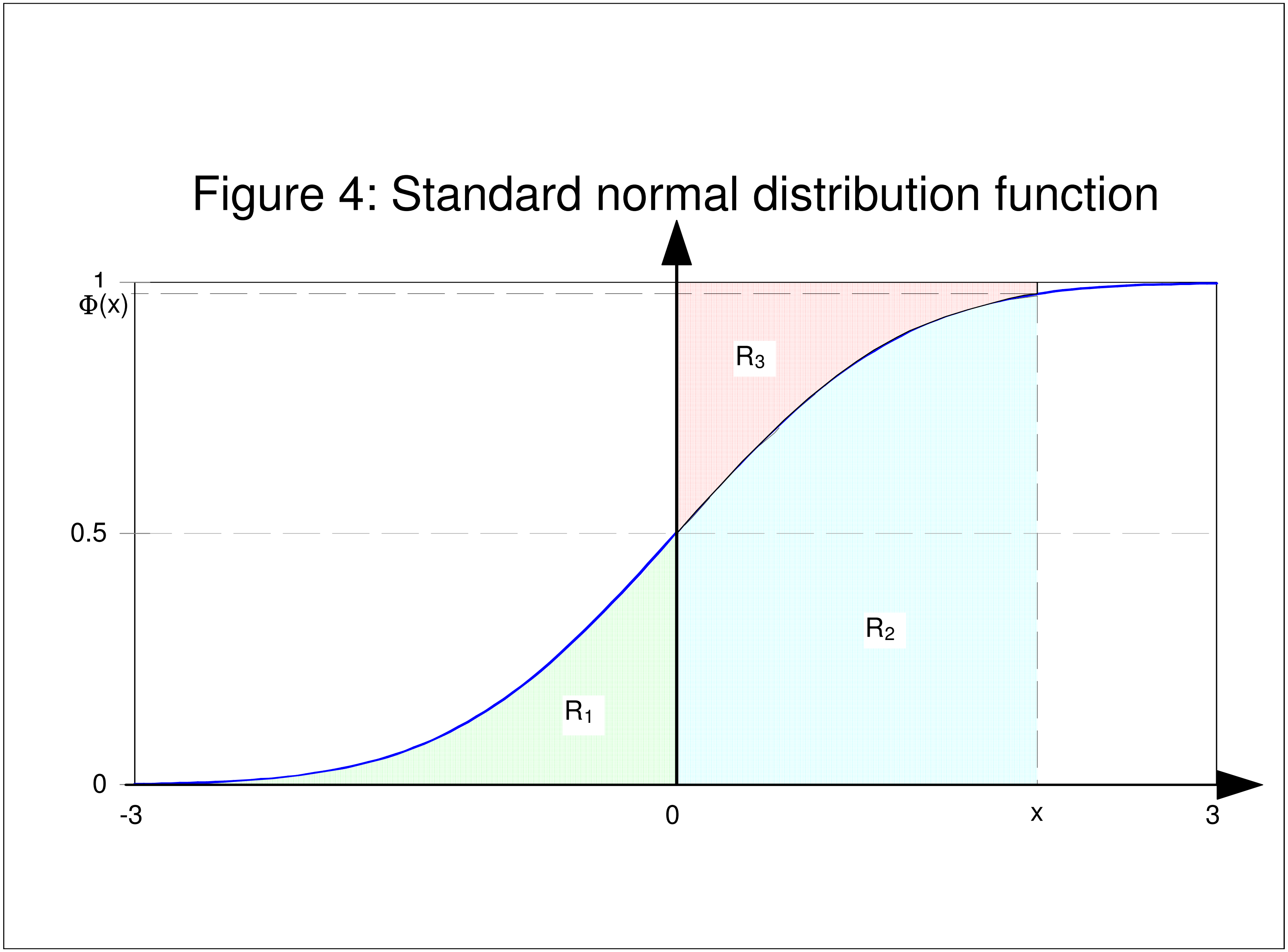


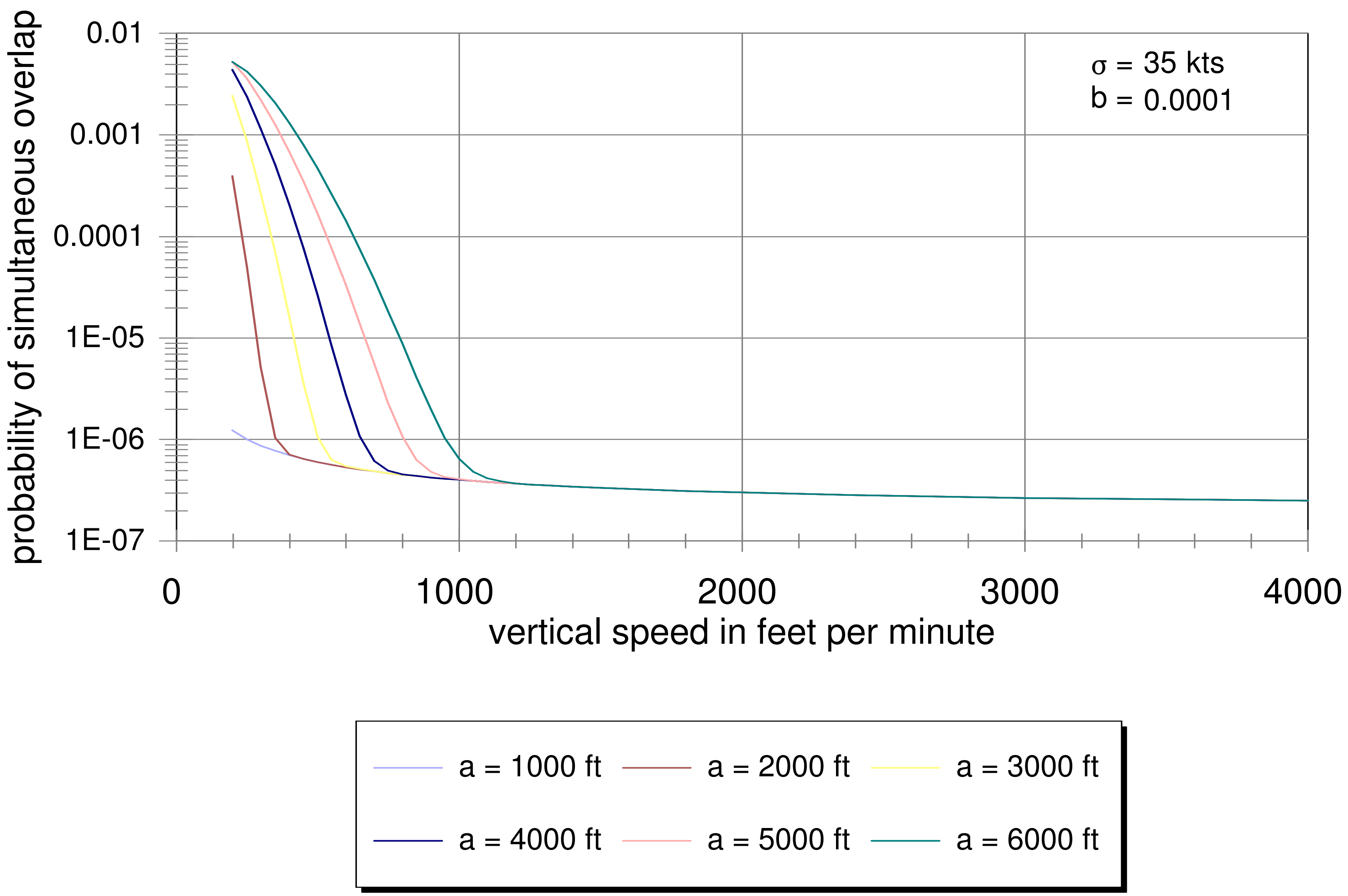
Figure 4: Standard normal distribution function
1
Φ(x)
0.5
0
-3
0
x
3
R3
R2
R1

Figure 5: Computation of $P(O)$

Computation of the probability of simultaneous longitudinal and vertical overlap

| | | | | |
|---|---|---|---|---|
| 1meter = | 3.2808399 | feet | | |
| 1 nmi = | 6076.1154856 | feet | | |
| $a$ = | 1000 | feet = | 0.1645788 nmi | |
| $h$ = | 65 | feet = | 0.0106976 nmi | |
| $c$ = | 200 | ft/min = | 1.9749460 kt | |
| $t_b$ = | $(a - h)/c$ | hr = | 0.0779167 hr | |
| $t_e$ = | $(a + h)/c$ | hr = | 0.08875 hr | |
| $l$ = | 0.03 | nmi | | |
| $m$ = | 30 | nmi | | |
| $k$ = | 0.5 | | | |
| $b$ = | 0.0001 | | | |
| $\sigma$ = | 35 | kt | | |
| | | | | |
| $\sigma t_b$ = | 2.727083333 | | | |
| $(m + l)/(\sigma t_b)$ = | 11.011764706 | <--sumarg | | |
| $\Psi[(m + l)/(\sigma t_b)]$ = | 11.011764706 | <--psisumarg | NA | <--tpsisumarg |
| $(km + l)/(\sigma t_b)$ = | 5.511382735 | <--sumargk | | |
| $\Psi[(km + l)/(\sigma t_b)]$ = | 5.511382738 | <--psisumargk | 5.511382738 | <--tpsisumargk |
| $\sigma t_e$ = | 3.106250000 | | | |
| $(m - l)/(\sigma t_e)$ = | 9.648289738 | <--difarg | | |
| $\Psi[(m - l)/(\sigma t_e)]$ = | 9.648289738 | <--psidifarg | NA | <--tpsidifarg |
| $(km - l)/(\sigma t_e)$ = | 4.819315895 | <--difargk | | |
| $\Psi[(km - l)/(\sigma t_e)]$ = | 4.819316034 | <--psidifargk | 4.819316034 | <--tpsidifargk |
| $(b - k)/(mk(1-k))$ = | -0.066653333 | | | |
| $(1 - b)/(m(1-k))$ = | 0.066660000 | | | |
| $b/(km)$ = | 0.000006667 | | | |
| $\Psi(0) = 1/\sqrt{(2\pi)}$ = | 0.398942280 | | | |
| $P(O)$ = | 1.24e-06 | <--po | | |

# Figure 6: Probability of Simultaneous Longitudinal and Vertical Overlap



## 4 Lateral Overlap

4.1 Airplanes that participate in an ITP are expected to navigate by using the global navigation satellite system (GNSS). Such airplanes ordinarily experience very small lateral deviations from the center lines of their planned routes of flight. However, in order to account for the possibility of operational errors, analysts have sometimes modeled a GNSS-equipped airplane's lateral deviation from its center line, at any randomly chosen moment, as a normal-double-exponential (NDE) random variable, typically called $Y$. The NDE density function is then a weighted sum of a normal density with mean 0 and standard deviation $\sigma_L$, and a double exponential density with parameter $1/\lambda$. The weighting parameter – a number in the interval [0, 1] – is usually called $\alpha$. Adapting a formula from reference 10.2 – for the special case in which planned lateral separation is 0 nmi – we find that if two airplanes with wingspan $w$ are assigned to the same route, and both airplanes' lateral deviations are described by the NDE density function

$$f_Y(y) = \frac{1-\alpha}{\sigma_L\sqrt{2\pi}}\, e^{-\frac{y^2}{2\sigma_L^2}} + \frac{\alpha}{2\lambda}\, e^{-\frac{|y|}{\lambda}},$$

then their lateral overlap probability, $p_0$, is given by

$$p_0 = (1-\alpha)^2\left[2\Phi\left(\frac{w}{\sqrt{2}\sigma_L}\right) - 1\right] + \alpha^2\left[1 - \frac{w+2\lambda}{2\lambda}\cdot e^{-\frac{w}{\lambda}}\right]$$
$$+ \; 2\alpha(1-\alpha)\left\{ e^{\frac{\sigma_L^2}{2\lambda^2}}\left[ e^{\frac{w}{\lambda}}\Phi\left(-\frac{w}{\sigma_L} - \frac{\sigma_L}{\lambda}\right) - e^{\frac{-w}{\lambda}}\Phi\left(\frac{w}{\sigma_L} - \frac{\sigma_L}{\lambda}\right)\right] + 2\Phi\left(\frac{w}{\sigma_L}\right) - 1\right\}. \qquad (5)$$

## 5 The probability of a nose-to-tail collision

5.1 Let $C_{NT}$ denote the event in which the aircraft experience a nose-to-tail collision. $C_{NT}$ occurs if and only if $a_1$ and $a_2$ enter into longitudinal overlap during a period in which they are already in vertical and lateral overlap. Therefore, if $E_3$ or $E_4$ occurs – i.e., if the airplanes are already in longitudinal overlap when $a_2$ begins its change of altitude – then a nose-to-tail collision cannot possibly occur. As was noted in paragraph 2.9, no longitudinal overlap occurs when $E_1$ or $E_6$ occurs; and thus, again, we conclude that a nose-to-tail collision is impossible. We are left to find the probability of a nose-to-tail collision when either $E_2$ or $E_5$ occurs.

5.2 The probability that the airplanes enter into longitudinal overlap during their period of vertical overlap is $P(t_b \le T_b \le t_e) = P(C_4 \cup C_5)$. Since the $E_i \cap C_j$ are mutually disjoint, the probability that one of $E_2$ or $E_5$ occurs, and that one of $C_4$ or $C_5$ also occurs, is $P([E_2 \cup E_5] \cap [C_4 \cup C_5])$ $= P(E_2 \cap C_4) + P(E_2 \cap C_5) + P(E_5 \cap C_4) + P(E_5 \cap C_5)$. Recalling results from paragraphs 3.3.3, 3.3.4, 3.6.3 and 3.6.4, we rewrite this sum of probabilities as

$$\iint_{A_{2,4}} f_{U,V}(u,v)\,dv\,du + \iint_{A_{2,5}} f_{U,V}(u,v)\,dv\,du + \iint_{A_{5,4}} f_{U,V}(u,v)\,dv\,du + \iint_{A_{5,5}} f_{U,V}(u,v)\,dv\,du$$

$$= \left[\int_{-m}^{-km}\int_{(l-u)/t_e}^{(-l-u)/t_b} \frac{1-b}{2m(1-k)} n(v;0,\sigma^2)\,dv\,du \;+\; \int_{-km}^{-al/h}\int_{(l-u)/t_e}^{(-l-u)/t_b} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du\right]$$

$$+ \left[\int_{-m}^{-km}\int_{(-l-u)/t_e}^{(l-u)/t_e} \frac{1-b}{2m(1-k)} n(v;0,\sigma^2)\,dv\,du \;+\; \int_{-km}^{-al/h}\int_{(-l-u)/t_e}^{(l-u)/t_e} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du \;+\; \int_{-al/h}^{-l}\int_{(-l-u)/t_e}^{(-l-u)/t_b} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du\right]$$

$$+ \left[\int_{al/h}^{km}\int_{(l-u)/t_b}^{(-l-u)/t_e} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du \;+\; \int_{km}^{m}\int_{(l-u)/t_b}^{(-l-u)/t_e} \frac{1-b}{2m(1-k)} n(v;0,\sigma^2)\,dv\,du\right]$$

$$+ \left[\int_{l}^{al/h}\int_{(l-u)/t_b}^{(l-u)/t_e} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du \;+\; \int_{al/h}^{km}\int_{(-l-u)/t_e}^{(l-u)/t_e} \frac{b}{2km} n(v;0,\sigma^2)\,dv\,du \;+\; \int_{km}^{m}\int_{(-l-u)/t_e}^{(l-u)/t_e} \frac{1-b}{2m(1-k)} n(v;0,\sigma^2)\,dv\,du\right]$$

$$= \frac{1-b}{2m(1-k)}\left[\int_{-m}^{-km}\int_{(-l-u)/t_e}^{(-l-u)/t_b} n(v;0,\sigma^2)\,dv\,du + \int_{km}^{m}\int_{(l-u)/t_b}^{(l-u)/t_e} n(v;0,\sigma^2)\,dv\,du\right]$$

$$+ \frac{b}{2km}\left[\int_{-km}^{-al/h}\int_{(-l-u)/t_e}^{(-l-u)/t_b} n(v;0,\sigma^2)\,dv\,du + \int_{-al/h}^{-l}\int_{(-l-u)/t_e}^{(-l-u)/t_b} n(v;0,\sigma^2)\,dv\,du\right]$$

$$+ \frac{b}{2km}\left[\int_{l}^{al/h}\int_{(l-u)/t_b}^{(l-u)/t_e} n(v;0,\sigma^2)\,dv\,du + \int_{al/h}^{km}\int_{(l-u)/t_b}^{(l-u)/t_e} n(v;0,\sigma^2)\,dv\,du\right]$$

$$= \frac{1-b}{2m(1-k)}\left[\int_{-m}^{-km}\int_{(-l-u)/t_e}^{(-l-u)/t_b} n(v;0,\sigma^2)\,dv\,du + \int_{km}^{m}\int_{(l-u)/t_b}^{(l-u)/t_e} n(v;0,\sigma^2)\,dv\,du\right]$$

$$+ \frac{b}{2km}\left[\int_{-km}^{-l}\int_{(-l-u)/t_e}^{(-l-u)/t_b} n(v;0,\sigma^2)\,dv\,du + \int_{l}^{km}\int_{(l-u)/t_b}^{(l-u)/t_e} n(v;0,\sigma^2)\,dv\,du\right]$$

$$= \frac{1-b}{2m(1-k)}\left[\int_{-m}^{-km}\left[\Phi\left(\frac{-l-u}{\sigma t_b}\right) - \Phi\left(\frac{-l-u}{\sigma t_e}\right)\right]du + \int_{km}^{m}\left[\Phi\left(\frac{l-u}{\sigma t_e}\right) - \Phi\left(\frac{l-u}{\sigma t_b}\right)\right]du\right]$$

$$+ \frac{b}{2km}\left[\int_{-km}^{-l}\left[\Phi\left(\frac{-l-u}{\sigma t_b}\right) - \Phi\left(\frac{-l-u}{\sigma t_e}\right)\right]du + \int_{l}^{km}\left[\Phi\left(\frac{l-u}{\sigma t_e}\right) - \Phi\left(\frac{l-u}{\sigma t_b}\right)\right]du\right]$$

$$= \frac{1-b}{2m(1-k)}\left[\int_{-m}^{-km}\Phi\left(\frac{-l-u}{\sigma t_b}\right)du - \int_{-m}^{-km}\Phi\left(\frac{-l-u}{\sigma t_e}\right)du + \int_{km}^{m}\Phi\left(\frac{l-u}{\sigma t_e}\right)du - \int_{km}^{m}\Phi\left(\frac{l-u}{\sigma t_b}\right)du\right]$$

$$+ \frac{b}{2km}\left[\int_{-km}^{-l}\Phi\left(\frac{-l-u}{\sigma t_b}\right)du - \int_{-km}^{-l}\Phi\left(\frac{-l-u}{\sigma t_e}\right)du + \int_{l}^{km}\Phi\left(\frac{l-u}{\sigma t_e}\right)du - \int_{l}^{km}\Phi\left(\frac{l-u}{\sigma t_b}\right)du\right].$$

5.3 As in paragraph 3.7.4 we let $w = \frac{l-u}{\sigma t_b}$, $x = \frac{-l-u}{\sigma t_e}$, $y = \frac{l-u}{\sigma t_e}$, and $z = \frac{-l-u}{\sigma t_b}$ ; and we then rewrite the probability of entry into longitudinal overlap during the period of vertical overlap as

$$\frac{1-b}{2m(1-k)}\left[\int_{\frac{m-l}{\sigma t_b}}^{\frac{km-l}{\sigma t_b}}\Phi(z)(-\sigma t_b dz) - \int_{\frac{m-l}{\sigma t_e}}^{\frac{km-l}{\sigma t_e}}\Phi(x)(-\sigma t_e dx) + \int_{-\frac{km-l}{\sigma t_e}}^{-\frac{m-l}{\sigma t_e}}\Phi(y)(-\sigma t_e dy) - \int_{-\frac{km-l}{\sigma t_b}}^{-\frac{m-l}{\sigma t_b}}\Phi(w)(-\sigma t_b dw)\right]$$

$$+ \frac{b}{2km}\left[\int_{\frac{km-l}{\sigma t_b}}^{0}\Phi(z)(-\sigma t_b dz) - \int_{\frac{km-l}{\sigma t_e}}^{0}\Phi(x)(-\sigma t_e dx) + \int_{0}^{-\frac{km-l}{\sigma t_e}}\Phi(y)(-\sigma t_e dy) - \int_{0}^{-\frac{km-l}{\sigma t_b}}\Phi(w)(-\sigma t_b dw)\right]$$

$$= \frac{1-b}{2m(1-k)}\left[\sigma t_b\left(\int_{\frac{km-l}{\sigma t_b}}^{\frac{m-l}{\sigma t_b}}\Phi(z)dz - \int_{-\frac{m-l}{\sigma t_b}}^{-\frac{km-l}{\sigma t_b}}\Phi(w)dw\right) + \sigma t_e\left(\int_{-\frac{m-l}{\sigma t_e}}^{-\frac{km-l}{\sigma t_e}}\Phi(y)dy - \int_{\frac{km-l}{\sigma t_e}}^{\frac{m-l}{\sigma t_e}}\Phi(x)dx\right)\right]$$

$$+ \frac{b}{2km}\left[\sigma t_b\left(\int_{0}^{\frac{km-l}{\sigma t_b}}\Phi(z)dz - \int_{-\frac{km-l}{\sigma t_b}}^{0}\Phi(w)dw\right) + \sigma t_e\left(\int_{-\frac{km-l}{\sigma t_e}}^{0}\Phi(y)dy - \int_{0}^{\frac{km-l}{\sigma t_e}}\Phi(x)dx\right)\right]$$

$$= \frac{1-b}{2m(1-k)}\left[\sigma t_b\left(\left[\Psi\left(\frac{m-l}{\sigma t_b}\right) - \Psi\left(\frac{km-l}{\sigma t_b}\right)\right] - \left[\Psi\left(-\frac{km-l}{\sigma t_b}\right) - \Psi\left(-\frac{m-l}{\sigma t_b}\right)\right]\right)\right.$$

$$\left. + \ \sigma t_e\left(\left[\Psi\left(-\frac{km-l}{\sigma t_e}\right) - \Psi\left(-\frac{m-l}{\sigma t_e}\right)\right] - \left[\Psi\left(\frac{m-l}{\sigma t_e}\right) - \Psi\left(\frac{km-l}{\sigma t_e}\right)\right]\right)\right]$$

$$+ \ \frac{b}{2km}\left[\sigma t_b\left(\left[\Psi\left(\frac{km-l}{\sigma t_b}\right) - \Psi(0)\right] - \left[\Psi(0) - \Psi\left(-\frac{km-l}{\sigma t_b}\right)\right]\right)\right.$$

$$\left. + \ \sigma t_e\left(\left[\Psi(0) - \Psi\left(-\frac{km-l}{\sigma t_e}\right)\right] - \left[\Psi\left(\frac{km-l}{\sigma t_e}\right) - \Psi(0)\right]\right)\right]$$

$$= \frac{1-b}{2m(1-k)}\left[\sigma t_b\left(\left[\Psi\left(\frac{m-l}{\sigma t_b}\right) + \Psi\left(-\frac{m-l}{\sigma t_b}\right)\right] - \left[\Psi\left(\frac{km-l}{\sigma t_b}\right) + \Psi\left(-\frac{km-l}{\sigma t_b}\right)\right]\right)\right.$$

$$\left. + \ \sigma t_e\left(\left[\Psi\left(\frac{km-l}{\sigma t_e}\right) + \Psi\left(-\frac{km-l}{\sigma t_e}\right)\right] - \left[\Psi\left(\frac{m-l}{\sigma t_e}\right) + \Psi\left(-\frac{m-l}{\sigma t_e}\right)\right]\right)\right]$$

$$+ \ \frac{b}{2km}\left[\sigma t_b\left(\left[\Psi\left(\frac{km-l}{\sigma t_b}\right) - \Psi(0)\right] + \left[\Psi\left(-\frac{km-l}{\sigma t_b}\right) - \Psi(0)\right]\right)\right.$$

$$\left. - \ \sigma t_e\left(\left[\Psi\left(\frac{km-l}{\sigma t_e}\right) - \Psi(0)\right] + \left[\Psi\left(-\frac{km-l}{\sigma t_e}\right) - \Psi(0)\right]\right)\right].$$

5.4 Applying equation (4c) to the first and second lines of this last expression, applying equation (4a) to its third and fourth lines, and remembering that $\Psi(0) = 1/\sqrt{2\pi}$, we write the probability of entry into longitudinal overlap during a period of vertical overlap, as $P([E_2 \cup E_5] \cap [C_4 \cup C_5]) = P(E_2 \cap C_4) + P(E_2 \cap C_5) + P(E_5 \cap C_4) + P(E_5 \cap C_5) =$

$$\frac{1-b}{2m(1-k)}\left\{\sigma t_b\left(\left[2\Psi\left(\frac{m-l}{\sigma t_b}\right) - \frac{m-l}{\sigma t_b}\right] - \left[2\Psi\left(\frac{km-l}{\sigma t_b}\right) - \frac{km-l}{\sigma t_b}\right]\right)\right.$$

$$\left. + \ \sigma t_e\left(\left[2\Psi\left(\frac{km-l}{\sigma t_e}\right) - \frac{km-l}{\sigma t_e}\right] - \left[2\Psi\left(\frac{m-l}{\sigma t_e}\right) - \frac{m-l}{\sigma t_e}\right]\right)\right\}$$

$$+ \ \frac{b}{2km}\left\{\sigma t_b\left(2\left[\Psi\left(\frac{km-l}{\sigma t_b}\right) - \Psi(0)\right] - \frac{km-l}{\sigma t_b}\right) - \sigma t_e\left(2\left[\Psi\left(\frac{km-l}{\sigma t_e}\right) - \Psi(0)\right] - \frac{km-l}{\sigma t_e}\right)\right\}$$

$$= \frac{1-b}{2m(1-k)} \left\{ 2\sigma t_b \Psi\left( \frac{m-l}{\sigma t_b} \right) - (m-l) - 2\sigma t_b \Psi\left( \frac{km-l}{\sigma t_b} \right) + (km-l) \right.$$

$$\left. + \; 2\sigma t_e \Psi\left( \frac{km-l}{\sigma t_e} \right) - (km-l) - 2\sigma t_e \Psi\left( \frac{m-l}{\sigma t_e} \right) + (m-l) \right\}$$

$$+ \; \frac{b}{2km} \left\{ 2\sigma t_b \Psi\left( \frac{km-l}{\sigma t_b} \right) - 2\sigma t_b \Psi(0) - (km-l) - 2\sigma t_e \Psi\left( \frac{km-l}{\sigma t_e} \right) + 2\sigma t_e \Psi(0) + (km-l) \right\}$$

$$= \frac{1-b}{2m(1-k)} \left\{ 2\sigma t_b \Psi\left( \frac{m-l}{\sigma t_b} \right) - 2\sigma t_b \Psi\left( \frac{km-l}{\sigma t_b} \right) + 2\sigma t_e \Psi\left( \frac{km-l}{\sigma t_e} \right) - 2\sigma t_e \Psi\left( \frac{m-l}{\sigma t_e} \right) \right\}$$

$$+ \; \frac{b}{2km} \left\{ 2\sigma t_b \Psi\left( \frac{km-l}{\sigma t_b} \right) - 2\sigma t_e \Psi\left( \frac{km-l}{\sigma t_e} \right) + 2\sigma (t_e - t_b) \Psi(0) \right\}$$

$$= \left[ \sigma t_b \Psi\left( \frac{km-l}{\sigma t_b} \right) - \sigma t_e \Psi\left( \frac{km-l}{\sigma t_e} \right) \right] \cdot \left[ \frac{b}{km} - \frac{1-b}{m(1-k)} \right]$$

$$+ \; \frac{1-b}{m(1-k)} \cdot \left[ \sigma t_b \Psi\left( \frac{m-l}{\sigma t_b} \right) - \sigma t_e \Psi\left( \frac{m-l}{\sigma t_e} \right) \right] + \frac{b}{km} \cdot \sigma (t_e - t_b) \Psi(0)$$

$$= \frac{b-k}{mk(1-k)} \cdot \left[ \sigma t_b \Psi\left( \frac{km-l}{\sigma t_b} \right) - \sigma t_e \Psi\left( \frac{km-l}{\sigma t_e} \right) \right]$$

$$+ \; \frac{1-b}{m(1-k)} \cdot \left[ \sigma t_b \Psi\left( \frac{m-l}{\sigma t_b} \right) - \sigma t_e \Psi\left( \frac{m-l}{\sigma t_e} \right) \right] + \frac{b}{mk} \cdot \frac{\sigma (t_e - t_b)}{\sqrt{2\pi}} \; . \tag{6a}$$

5.5 If all arguments of $\Psi$ in equation (6a) are greater than 6, we can safely apply the approximation $\Psi(x) \approx x$ to the right side of the equation, and write $P([E_2 \cup E_5] \cap [C_4 \cup C_5]) \approx$

$$\frac{b-k}{mk(1-k)} \cdot \left[ \sigma t_b \cdot \frac{km-l}{\sigma t_b} - \sigma t_e \cdot \frac{km-l}{\sigma t_e} \right] + \frac{1-b}{m(1-k)} \cdot \left[ \sigma t_b \cdot \frac{m-l}{\sigma t_b} - \sigma t_e \cdot \frac{m-l}{\sigma t_e} \right] + \frac{b}{mk} \cdot \frac{\sigma (t_e - t_b)}{\sqrt{2\pi}}$$

$$= \frac{b}{mk} \cdot \frac{\sigma (t_e - t_b)}{\sqrt{2\pi}} \; . \tag{6b}$$

5.6 The probability that $a_1$ and $a_2$ are in lateral overlap at the instant when they enter into longitudinal overlap is $p_0$, because that is the probability that they are in lateral overlap at *any* randomly chosen instant. Thus $P(C_{NT}) = [P(E_2 \cap C_4) + P(E_2 \cap C_5) + P(E_5 \cap C_4) + P(E_5 \cap C_5)] \cdot p_0$

$$= \left\{ \frac{b-k}{mk(1-k)} \cdot \left[ \sigma t_b \Psi\left( \frac{km-l}{\sigma t_b} \right) - \sigma t_e \Psi\left( \frac{km-l}{\sigma t_e} \right) \right] \right.$$

$$\left. + \frac{1-b}{m(1-k)} \cdot \left[ \sigma t_b \Psi\left( \frac{m-l}{\sigma t_b} \right) - \sigma t_e \Psi\left( \frac{m-l}{\sigma t_e} \right) \right] + \frac{b}{mk} \cdot \frac{\sigma(t_e - t_b)}{\sqrt{2\pi}} \right\} \cdot p_0 \; ; \qquad (7a)$$

and when all arguments of $\Psi$ in equation (7a) are greater than 6, we can safely approximate

$$P(C_{NT}) \approx \frac{b}{mk} \cdot \frac{\sigma(t_e - t_b)}{\sqrt{2\pi}} \cdot p_0 \, . \qquad (7b)$$

## 6 The probability of a top-to-bottom collision

6.1 Aircraft $a_1$ and $a_2$ experience a top-to-bottom collision if and only if they enter into vertical overlap during a period in which they are already in longitudinal and lateral overlap. If $E_1$ or $E_6$ occurs, there is no possibility of a longitudinal overlap, and, therefore, no possibility of a top-to-bottom collision. We are left to find the probability of a top-to-bottom collision when $E_2$, $E_3$, $E_4$ or $E_5$ occurs.

6.2 The probability that the airplanes enter into vertical overlap during a period of longitudinal overlap is $P(T_b \leq t_b \leq T_e) = P(C_2 \cup C_3)$. Since the $E_i \cap C_j$ are mutually disjoint, the probability that one of $E_2$, $E_3$, $E_4$ or $E_5$ occurs, and that one of $C_2$ or $C_3$ also occurs, is $P([E_2 \cup E_3 \cup E_4 \cup E_5] \cap [C_2 \cup C_3]) =$

$$P(E_2 \cap C_2) + P(E_2 \cap C_3) + P(E_3 \cap C_2) + P(E_3 \cap C_3)$$
$$+ P(E_4 \cap C_2) + P(E_4 \cap C_3) + P(E_5 \cap C_2) + P(E_5 \cap C_3).$$

Recalling results from paragraphs 3.3.1, 3.3.2, 3.4.1, 3.4.2, 3.5.1, 3.5.2, 3.6.1 and 3.6.2, we rewrite this sum as

$$\iint_{A_{2,2}} f_{U,V}(u,v)\,dv\,du + \iint_{A_{2,3}} f_{U,V}(u,v)\,dv\,du + \iint_{A_{3,2}} f_{U,V}(u,v)\,dv\,du + \iint_{A_{3,3}} f_{U,V}(u,v)\,dv\,du$$
$$+ \iint_{A_{4,2}} f_{U,V}(u,v)\,dv\,du + \iint_{A_{4,3}} f_{U,V}(u,v)\,dv\,du + \iint_{A_{5,2}} f_{U,V}(u,v)\,dv\,du + \iint_{A_{5,3}} f_{U,V}(u,v)\,dv\,du$$

$$= \iint_{A_{2,2} \cup A_{2,3}} f_{U,V}(u,v)\,dv\,du + \iint_{A_{3,2} \cup A_{3,3} \cup A_{4,2} \cup A_{4,3}} f_{U,V}(u,v)\,dv\,du + \iint_{A_{5,2} \cup A_{5,3}} f_{U,V}(u,v)\,dv\,du$$

$$= \int_{-m}^{-km} \int_{(-l-u)/t_b}^{(l-u)/t_b} \frac{1-b}{2m(1-k)} \cdot n(v;0,\sigma^2)\,dv\,du + \int_{-km}^{-l} \int_{(-l-u)/t_b}^{(l-u)/t_b} \frac{b}{2km} \cdot n(v;0,\sigma^2)\,dv\,du$$

$$+ \int_{-l}^{l} \int_{(-l-u)/t_b}^{(l-u)/t_b} \frac{b}{2km} \cdot n(v;0,\sigma^2)\,dv\,du$$

$$+ \int_{l}^{km} \int_{(-l-u)/t_b}^{(l-u)/t_b} \frac{b}{2km} \cdot n(v;0,\sigma^2)\,dv\,du + \int_{km}^{m} \int_{(-l-u)/t_b}^{(l-u)/t_b} \frac{1-b}{2m(1-k)} \cdot n(v;0,\sigma^2)\,dv\,du$$

$$= \frac{1-b}{2m(1-k)} \left[ \int_{-m}^{-km} \int_{(-l-u)/t_b}^{(l-u)/t_b} n(v;0,\sigma^2)\,dv\,du + \int_{km}^{m} \int_{(-l-u)/t_b}^{(l-u)/t_b} n(v;0,\sigma^2)\,dv\,du \right]$$

$$+ \frac{b}{2km} \left[ \int_{-km}^{-l} \int_{(-l-u)/t_b}^{(l-u)/t_b} n(v;0,\sigma^2)\,dv\,du + \int_{-l}^{l} \int_{(-l-u)/t_b}^{(l-u)/t_b} n(v;0,\sigma^2)\,dv\,du + \int_{l}^{km} \int_{(-l-u)/t_b}^{(l-u)/t_b} n(v;0,\sigma^2)\,dv\,du \right]$$

$$= \frac{1-b}{2m(1-k)} \left\{ \int_{-m}^{-km} \left[ \Phi\left(\frac{l-u}{\sigma t_b}\right) - \Phi\left(\frac{-l-u}{\sigma t_b}\right) \right] du + \int_{km}^{m} \left[ \Phi\left(\frac{l-u}{\sigma t_b}\right) - \Phi\left(\frac{-l-u}{\sigma t_b}\right) \right] du \right\}$$

$$+ \frac{b}{2km} \left\{ \int_{-km}^{km} \left[ \Phi\left(\frac{l-u}{\sigma t_b}\right) - \Phi\left(\frac{-l-u}{\sigma t_b}\right) \right] du \right\}$$

$$= \frac{1-b}{2m(1-k)} \left[ \int_{-m}^{-km} \Phi\left(\frac{l-u}{\sigma t_b}\right) du - \int_{-m}^{-km} \Phi\left(\frac{-l-u}{\sigma t_b}\right) du + \int_{km}^{m} \Phi\left(\frac{l-u}{\sigma t_b}\right) du - \int_{km}^{m} \Phi\left(\frac{-l-u}{\sigma t_b}\right) du \right]$$

$$+ \frac{b}{2km} \left[ \int_{-km}^{km} \Phi\left(\frac{l-u}{\sigma t_b}\right) du - \int_{-km}^{km} \Phi\left(\frac{-l-u}{\sigma t_b}\right) du \right].$$

6.3 As in paragraph 3.7.4 we let $w = \dfrac{l-u}{\sigma t_b}$ and $z = \dfrac{-l-u}{\sigma t_b}$ ; and we then rewrite the probability of entry into vertical overlap during a period of longitudinal overlap as

$$\frac{1-b}{2m(1-k)} \left[ \int_{\frac{m+l}{\sigma t_b}}^{\frac{km+l}{\sigma t_b}} \Phi(w)(-\sigma t_b dw) - \int_{\frac{m-l}{\sigma t_b}}^{\frac{km-l}{\sigma t_b}} \Phi(z)(-\sigma t_b dz) + \int_{-\frac{km-l}{\sigma t_b}}^{-\frac{m-l}{\sigma t_b}} \Phi(w)(-\sigma t_b dw) - \int_{-\frac{km+l}{\sigma t_b}}^{-\frac{m+l}{\sigma t_b}} \Phi(z)(-\sigma t_b dz) \right]$$

$$+ \frac{b}{2km} \left[ \int_{\frac{km+l}{\sigma t_b}}^{-\frac{km-l}{\sigma t_b}} \Phi(w)(-\sigma t_b dw) - \int_{\frac{km-l}{\sigma t_b}}^{-\frac{km+l}{\sigma t_b}} \Phi(z)(-\sigma t_b dz) \right]$$

$$= \frac{(1-b)\sigma t_b}{2m(1-k)}\left[\int_{\frac{km+l}{\sigma t_b}}^{\frac{m+l}{\sigma t_b}}\Phi(w)dw - \int_{\frac{km-l}{\sigma t_b}}^{\frac{m-l}{\sigma t_b}}\Phi(z)dz + \int_{-\frac{m-l}{\sigma t_b}}^{-\frac{km-l}{\sigma t_b}}\Phi(w)dw - \int_{-\frac{m+l}{\sigma t_b}}^{-\frac{km+l}{\sigma t_b}}\Phi(z)dz\right]$$
$$+ \frac{b\sigma t_b}{2km}\left[\int_{-\frac{km-l}{\sigma t_b}}^{\frac{km+l}{\sigma t_b}}\Phi(w)dw - \int_{-\frac{km+l}{\sigma t_b}}^{\frac{km-l}{\sigma t_b}}\Phi(z)dz\right]$$

$$= \frac{(1-b)\sigma t_b}{2m(1-k)}\left[\Psi\left(\frac{m+l}{\sigma t_b}\right) - \Psi\left(\frac{km+l}{\sigma t_b}\right) - \left(\Psi\left(\frac{m-l}{\sigma t_b}\right) - \Psi\left(\frac{km-l}{\sigma t_b}\right)\right)\right.$$
$$\left. + \Psi\left(-\frac{km-l}{\sigma t_b}\right) - \Psi\left(-\frac{m-l}{\sigma t_b}\right) - \left(\Psi\left(-\frac{km+l}{\sigma t_b}\right) - \Psi\left(-\frac{m+l}{\sigma t_b}\right)\right)\right]$$
$$+ \frac{b\sigma t_b}{2km}\left[\Psi\left(\frac{km+l}{\sigma t_b}\right) - \Psi\left(-\frac{km-l}{\sigma t_b}\right) - \left(\Psi\left(\frac{km-l}{\sigma t_b}\right) - \Psi\left(-\frac{km+l}{\sigma t_b}\right)\right)\right]$$

$$= \frac{(1-b)\sigma t_b}{2m(1-k)}\left[\Psi\left(\frac{m+l}{\sigma t_b}\right) + \Psi\left(-\frac{m+l}{\sigma t_b}\right) - \left(\Psi\left(\frac{km+l}{\sigma t_b}\right) + \Psi\left(-\frac{km+l}{\sigma t_b}\right)\right)\right.$$
$$\left. + \Psi\left(\frac{km-l}{\sigma t_b}\right) + \Psi\left(-\frac{km-l}{\sigma t_b}\right) - \left(\Psi\left(\frac{m-l}{\sigma t_b}\right) + \Psi\left(-\frac{m-l}{\sigma t_b}\right)\right)\right]$$
$$+ \frac{b\sigma t_b}{2km}\left[\Psi\left(\frac{km+l}{\sigma t_b}\right) + \Psi\left(-\frac{km+l}{\sigma t_b}\right) - \left(\Psi\left(\frac{km-l}{\sigma t_b}\right) + \Psi\left(-\frac{km-l}{\sigma t_b}\right)\right)\right].$$

6.4 Again applying equation (4c) we write the probability of entry into vertical overlap during a period of longitudinal overlap as

$$\frac{(1-b)\sigma t_b}{2m(1-k)}\left[2\Psi\left(\frac{m+l}{\sigma t_b}\right) - \frac{m+l}{\sigma t_b} - \left(2\Psi\left(\frac{km+l}{\sigma t_b}\right) - \frac{km+l}{\sigma t_b}\right)\right.$$
$$\left. + 2\Psi\left(\frac{km-l}{\sigma t_b}\right) - \frac{km-l}{\sigma t_b} - \left(2\Psi\left(\frac{m-l}{\sigma t_b}\right) - \frac{m-l}{\sigma t_b}\right)\right]$$
$$+ \frac{b\sigma t_b}{2km}\left[2\Psi\left(\frac{km+l}{\sigma t_b}\right) - \frac{km+l}{\sigma t_b} - \left(2\Psi\left(\frac{km-l}{\sigma t_b}\right) - \frac{km-l}{\sigma t_b}\right)\right]$$

$$= \frac{1-b}{2m(1-k)}\left[2\sigma t_b\Psi\left(\frac{m+l}{\sigma t_b}\right) - (m+l) - 2\sigma t_b\Psi\left(\frac{km+l}{\sigma t_b}\right) + (km+l)\right.$$

$$\left. + 2\sigma t_b\Psi\left(\frac{km-l}{\sigma t_b}\right) - (km-l) - 2\sigma t_b\Psi\left(\frac{m-l}{\sigma t_b}\right) + (m-l)\right]$$

$$+ \frac{b}{2km}\left[2\sigma t_b\Psi\left(\frac{km+l}{\sigma t_b}\right) - (km+l) - 2\sigma t_b\Psi\left(\frac{km-l}{\sigma t_b}\right) + (km-l)\right]$$

$$= \frac{1-b}{2m(1-k)}\left[2\sigma t_b\Psi\left(\frac{m+l}{\sigma t_b}\right) - 2\sigma t_b\Psi\left(\frac{km+l}{\sigma t_b}\right) + 2\sigma t_b\Psi\left(\frac{km-l}{\sigma t_b}\right) - 2\sigma t_b\Psi\left(\frac{m-l}{\sigma t_b}\right)\right]$$

$$+ \frac{b}{2km}\left[2\sigma t_b\Psi\left(\frac{km+l}{\sigma t_b}\right) - 2\sigma t_b\Psi\left(\frac{km-l}{\sigma t_b}\right) - 2l\right]$$

$$= \left[\sigma t_b\Psi\left(\frac{km+l}{\sigma t_b}\right) - \sigma t_b\Psi\left(\frac{km-l}{\sigma t_b}\right)\right]\cdot\left[\frac{b}{km} - \frac{1-b}{m(1-k)}\right]$$

$$+ \frac{1-b}{m(1-k)}\left[\sigma t_b\Psi\left(\frac{m+l}{\sigma t_b}\right) - \sigma t_b\Psi\left(\frac{m-l}{\sigma t_b}\right)\right] - \frac{bl}{km}$$

$$= \frac{b-k}{mk(1-k)}\left[\sigma t_b\Psi\left(\frac{km+l}{\sigma t_b}\right) - \sigma t_b\Psi\left(\frac{km-l}{\sigma t_b}\right)\right] + \frac{(1-b)}{m(1-k)}\left[\sigma t_b\Psi\left(\frac{m+l}{\sigma t_b}\right) - \sigma t_b\Psi\left(\frac{m-l}{\sigma t_b}\right)\right] - \frac{bl}{mk}\ . \quad (8a)$$

6.5 If all arguments of $\Psi$ in equation (8a) are greater than 6, we can safely apply the approximation $\Psi(x) \approx x$ to the right side of the equation, and write $P([E_2 \cup E_3 \cup E_4 \cup E_5] \cap [C_2 \cup C_3]) \approx$

$$\frac{b-k}{mk(1-k)}\left[\sigma t_b\frac{km+l}{\sigma t_b} - \sigma t_b\frac{km-l}{\sigma t_b}\right] + \frac{1-b}{m(1-k)}\left[\sigma t_b\frac{m+l}{\sigma t_b} - \sigma t_b\frac{m-l}{\sigma t_b}\right] - \frac{bl}{mk}$$

$$= \frac{2l(b-k)}{mk(1-k)} + \frac{2l(1-b)}{m(1-k)} - \frac{bl}{km} = \frac{2l(b-k)}{mk(1-k)} + \frac{2lk(1-b)}{mk(1-k)} - \frac{bl(1-k)}{mk(1-k)}$$

$$= \frac{2bl-2lk+2lk-2blk-bl+blk}{mk(1-k)} = \frac{bl-blk}{mk(1-k)} = \frac{bl}{mk}\ . \quad (8b)$$

6.6 Let $C_{TB}$ denote the event in which $a_1$ and $a_2$ experience a top-to-bottom collision. The probability that $a_1$ and $a_2$ are in lateral overlap at the instant when they enter into vertical overlap is $p_0$, because that is the probability that they are in lateral overlap at *any* randomly chosen instant.

Using equation (8a) we conclude that $P(C_{TB}) = P([E_2 \cup E_3 \cup E_4 \cup E_5] \cap [C_2 \cup C_3]) \cdot p_0 =$

$$\left\{ \frac{(b-k)\sigma t_b}{mk(1-k)} \left[ \Psi\left( \frac{km+l}{\sigma t_b} \right) - \Psi\left( \frac{km-l}{\sigma t_b} \right) \right] + \frac{(1-b)\sigma t_b}{m(1-k)} \left[ \Psi\left( \frac{m+l}{\sigma t_b} \right) - \Psi\left( \frac{m-l}{\sigma t_b} \right) \right] - \frac{bl}{mk} \right\} \cdot p_0 . \tag{9a}$$

When all arguments of $\Psi$ in equation (9a) are greater than 6, we can safely use formula (8b) instead of (8a), and approximate $P(C_{TB}) \approx \dfrac{bl}{mk} \cdot p_0$. (9b)

6.7 If $a_1$ and $a_2$ experience a simultaneous longitudinal and vertical overlap, that event can occur in exactly one of two possible ways: either the airplanes enter into longitudinal overlap when they are already in vertical overlap, or they enter into vertical overlap when they are already in longitudinal overlap. Equations (6a) and (6b) give us the probability of entry into longitudinal overlap during a period of vertical overlap; and equations (8a) and (8b) give the probability of entry into vertical overlap during a period of longitudinal overlap. Adding the right sides of equations (6a) and (8a), we find that the probability of a simultaneous overlap is

$$\begin{aligned}
&\frac{b-k}{mk(1-k)} \cdot \left[ \sigma t_b \Psi\left( \frac{km-l}{\sigma t_b} \right) - \sigma t_e \Psi\left( \frac{km-l}{\sigma t_e} \right) \right] \\
&\qquad + \frac{1-b}{m(1-k)} \cdot \left[ \sigma t_b \Psi\left( \frac{m-l}{\sigma t_b} \right) - \sigma t_e \Psi\left( \frac{m-l}{\sigma t_e} \right) \right] + \frac{b}{mk} \cdot \frac{\sigma(t_e - t_b)}{\sqrt{2\pi}} \\
&+ \frac{b-k}{mk(1-k)} \left[ \sigma t_b \Psi\left( \frac{km+l}{\sigma t_b} \right) - \sigma t_b \Psi\left( \frac{km-l}{\sigma t_b} \right) \right] + \frac{1-b}{m(1-k)} \left[ \sigma t_b \Psi\left( \frac{m+l}{\sigma t_b} \right) - \sigma t_b \Psi\left( \frac{m-l}{\sigma t_b} \right) \right] - \frac{bl}{mk} \\
&= \frac{b-k}{mk(1-k)} \left[ \sigma t_b \Psi\left( \frac{km+l}{\sigma t_b} \right) - \sigma t_e \Psi\left( \frac{km-l}{\sigma t_e} \right) \right] \\
&\qquad + \frac{1-b}{m(1-k)} \left[ \sigma t_b \Psi\left( \frac{m+l}{\sigma t_b} \right) - \sigma t_e \Psi\left( \frac{m-l}{\sigma t_e} \right) \right] + \frac{b}{mk} \left[ \frac{\sigma(t_e - t_b)}{\sqrt{2\pi}} - l \right].
\end{aligned}$$

Since this last expression is exactly the same as the right side of equation (3c), we have just shown that the results of this section and the preceding section are consistent with those of section 3.

## 7 The probability of a side-to-side collision

7.1 Recall that airplanes $a_1$ and $a_2$ are assigned to the same route, and $p_0$ denotes the probability that, at any randomly chosen moment during their flights, they have laterally overlapping positions. We now let $n_0$ denote the average rate, in occurrences per hour, at which such airplanes enter into lateral overlap.

7.2 Let $k_0$ kts denote the average lateral passing speed of airplanes that are assigned to the same route. Then $2w/k_0$ hrs is the average duration of a lateral overlap, i.e., the average time that such airplanes spend in overlap when they pass each other laterally. Since $2w/k_0$ is a relatively small number, we can approximate $p_0 \approx n_0 \cdot (2w/k_0)$, from which it immediately follows that $n_0 \approx (p_0 \cdot k_0)/(2w)$. Since the airplanes that participate in an ITP are expected to navigate by using the GNSS, we adopt an estimate of $k_0$ from reference 10.4.

7.3 Viewing entries into lateral overlap as randomly occurring events in a Poisson process, we recall that the probability of such an event during a period lasting τ hours, is $1 - e^{-n_0\tau}$. We take the period to be the interval during which the airplanes are simultaneously in longitudinal and vertical overlap; and since that is a small fraction of an hour, and $n_0$ is also a small number, we can be confident that $n_0\tau$ is small. Therefore, it makes sense to approximate the exponential function by the first two terms of its Taylor series expansion, and take $1 - e^{-n_0\tau} \approx 1 - [1 - n_0\tau] = n_0\tau$.

7.4 Airplanes $a_1$ and $a_2$ experience a side-to-side collision if and only if they enter into lateral overlap during a period in which they are already in both longitudinal and vertical overlap. Having estimated a value for $n_0$, we next estimate the average duration of a simultaneous longitudinal and vertical overlap in order to compute the probability of a side-to-side collision.

7.5 As we see from the table in paragraph 3.2, the duration of the period of simultaneous overlap may be a random variable. Indeed, that is the case whenever $C_2$, $C_4$, or $C_5$ occurs. When $C_2$ occurs, the period lasts $T_e - t_b$ hrs; when $C_4$ occurs, it lasts $T_e - T_b$ hrs; and when $C_5$ occurs, it lasts $t_e - T_b$ hrs. The duration of simultaneous overlap is the constant $t_e - t_b$ hrs only when $C_3$ occurs.

7.6 The following table gives the duration of simultaneous longitudinal and vertical overlap for each of the twelve events $E_i \cap C_j$ for which such overlap occurs.

| | | $C_j$: | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
|---|---|---|---|---|---|---|
| $E_i$ | $T_b$ | $T_e$ | $T_e - t_b$ | $t_e - t_b$ | $T_e - T_b$ | $t_e - T_b$ |
| $E_2$ | $\frac{-l-U}{V}$ | $\frac{l-U}{V}$ | $\frac{l-U}{V} - t_b$ | $t_e - t_b$ | $\frac{l-U}{V} - \frac{-l-U}{V} = \frac{2l}{V}$ | $t_e - \frac{-l-U}{V}$ |
| $E_3$ | 0 | $\frac{-l-U}{V}$ | $\frac{-l-U}{V} - t_b$ | $t_e - t_b$ | (no simultaneous overlap) | (no simultaneous overlap) |
| $E_4$ | 0 | $\frac{l-U}{V}$ | $\frac{l-U}{V} - t_b$ | $t_e - t_b$ | (no simultaneous overlap) | (no simultaneous overlap) |
| $E_5$ | $\frac{l-U}{V}$ | $\frac{-l-U}{V}$ | $\frac{-l-U}{V} - t_b$ | $t_e - t_b$ | $\frac{-l-U}{V} - \frac{l-U}{V} = -\frac{2l}{V}$ | $t_e - \frac{l-U}{V}$ |

For example, when $E_2$ occurs, $T_e = \dfrac{l-U}{V}$; and when $C_2$ occurs, the duration of the overlap interval is $T_e - t_b$. Therefore, when $E_2 \cap C_2$ occurs, the overlap duration is $\dfrac{l-U}{V} - t_b$.

7.7 We recall that $E_i \cap C_j$ occurs if and only if the random vector ($U$, $V$) takes a value in the set $A_{i,j}$; and we let $g(u,v)$ denote the duration of the simultaneous longitudinal and vertical overlap when ($U$, $V$) assumes any particular value ($u$, $v$). Then

$$g(u,v) = \begin{cases} \dfrac{l-u}{v} - t_b & \text{if } (u,v) \in A_{2,2} \cup A_{4,2} \\ \dfrac{-l-u}{v} - t_b & \text{if } (u,v) \in A_{3,2} \cup A_{5,2} \\ t_e - t_b & \text{if } (u,v) \in A_{2,3} \cup A_{3,3} \cup A_{4,3} \cup A_{5,3} \\ \dfrac{2l}{v} & \text{if } (u,v) \in A_{2,4} \\ -\dfrac{2l}{v} & \text{if } (u,v) \in A_{5,4} \\ t_e - \dfrac{-l-u}{v} & \text{if } (u,v) \in A_{2,5} \\ t_e - \dfrac{l-u}{v} & \text{if } (u,v) \in A_{5,5} \\ 0 & \text{otherwise}. \end{cases}$$

We then estimate the unconditional mean value of the duration of simultaneous longitudinal and vertical overlap by $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v) f_{U,V}(u,v)\,dv\,du$ =

$$\sum_{\substack{i=2,\dots,5 \\ j=2,3}} \iint_{A_{i,j}} g(u,v) f_{U,V}(u,v)\,dv\,du + \sum_{\substack{i=2,5 \\ j=4,5}} \iint_{A_{i,j}} g(u,v) f_{U,V}(u,v)\,dv\,du.$$

This unconditional average duration accounts for *all* possible values of ($U$, $V$), including those – nearly all of them – for which the duration is 0 – i.e., those arising from ITP executions in which no simultaneous overlap occurs! However, in estimating the probability of a side-to-side collision, we are not concerned with the unconditional mean value of the duration of simultaneous overlap, but rather with the mean duration of those overlaps that *do* occur. Therefore, we divide the unconditional mean value by the probability of simultaneous longitudinal and vertical overlap, and estimate the conditional mean duration of simultaneous overlap by

$$\frac{1}{P(O)}\left\{\sum_{\substack{i=2,\dots,5\\ j=2,3}}\iint_{A_{i,j}} g(u,v)\,f_{U,V}(u,v)\,dv\,du + \sum_{\substack{i=2,5\\ j=4,5}}\iint_{A_{i,j}} g(u,v)\,f_{U,V}(u,v)\,dv\,du\right\}. \qquad (10)$$

7.8 Though it is possible to evaluate all of the integrals in formula (10), we save a great deal of effort by adopting a very simple – and conservative – estimate for the mean duration of a simultaneous longitudinal and vertical overlap. We first note that the duration of the simultaneous overlap cannot exceed the duration of either of its "component" overlaps. That is, the duration of the simultaneous overlap is necessarily less than or equal to $T_e - T_b$ (the duration of the longitudinal overlap), and is necessarily less than or equal to $t_e - t_b$ (the duration of the vertical overlap). Since the simplifying assumptions of paragraph 2.10 give us a constant duration for the period of vertical overlap, we greatly simplify the computation by substituting that value for $g(u,v)$ in all eight of the integrals of formula (10) where it doesn't already appear. Since $g(u,v) \le t_e - t_b$ for all $(u,v)$ in the relevant sets $A_{i,j}$, we can be sure that we thereby obtain an overestimate of the time period in which a side-to-side collision may occur. That is,

$$\begin{aligned}
&\frac{1}{P(O)}\left\{\sum_{\substack{i=2,\dots,5\\ j=2,3}}\iint_{A_{i,j}} g(u,v)\,f_{U,V}(u,v)\,dv\,du + \sum_{\substack{i=2,5\\ j=4,5}}\iint_{A_{i,j}} g(u,v)\,f_{U,V}(u,v)\,dv\,du\right\}\\
&\le \frac{1}{P(O)}\left\{\sum_{\substack{i=2,\dots,5\\ j=2,3}}\iint_{A_{i,j}} (t_e-t_b)\,f_{U,V}(u,v)\,dv\,du + \sum_{\substack{i=2,5\\ j=4,5}}\iint_{A_{i,j}} (t_e-t_b)\,f_{U,V}(u,v)\,dv\,du\right\}\\
&= \frac{t_e-t_b}{P(O)}\left\{\sum_{\substack{i=2,\dots,5\\ j=2,3}}\iint_{A_{i,j}} f_{U,V}(u,v)\,dv\,du + \sum_{\substack{i=2,5\\ j=4,5}}\iint_{A_{i,j}} f_{U,V}(u,v)\,dv\,du\right\}\\
&= \frac{t_e-t_b}{P(O)}\cdot P(O) \;=\; t_e-t_b \;=\; 2h/c.
\end{aligned}$$

7.9 Let $C_{SS}$ denote the event in which $a_1$ and $a_2$ experience a side-to-side collision. The probability of a side-to-side collision is the probability that a simultaneous longitudinal and vertical overlap occurs, and that an entry into lateral overlap occurs during that simultaneous longitudinal and vertical overlap. Since the Reich model assumes that an aircraft's movement in each dimension is independent of its movement in the other two dimensions, the entry into lateral overlap is independent of the occurrence of a simultaneous longitudinal and vertical overlap. We estimate the probability of entry into lateral overlap, during *any* period of duration $t_e - t_b = 2h/c$, to be $n_0\cdot(t_e - t_b) = 2n_0 h/c$. Using approximation (3d) to the probability of simultaneous longitudinal and vertical overlap, we write $\mathrm{P}(C_{SS}) \approx n_0\cdot(t_e-t_b)\cdot \mathrm{P}(O) \approx \frac{2n_0 h}{c}\cdot\frac{b}{mk}\left(l + \frac{\sigma(t_e-t_b)}{\sqrt{2\pi}}\right)$. (11)

7.10 At the risk of mixing data from different sources, we cite parameter values from references 10.3 and 10.4 in order to illustrate the computation of a value for $n_0$. Letting $w$ = 0.032 nmi, $\alpha = 0.00564$, $\sigma_L = 0.0232$ nmi, and $\lambda = 0.038$ nmi – all of which are empirically derived values cited in reference 10.3 – equation (5) (of this report) yields $p_0 = 0.6686$. Following the method described in reference 10.4, we find that if $a_1$ and $a_2$ are both navigating by GNSS, their relative cross-track speed is $k_0 = \sqrt{2}\cdot 0.7838 = 1.10846$ (kts). We then compute $n_0 \approx (p_0 \cdot k_0)/(2w)$ = 11.58 (occurrences per hour).

## 8 Collision probability

8.1 A basic principle of the model presented in this paper is that airplanes can collide in only one of three ways: nose-to-tail, top-to-bottom, or side-to-side. Letting $C$ denote the event that airplanes $a_1$ and $a_2$ collide, we write $P(C) = P(C_{NT}) + P(C_{TB}) + P(C_{SS})$. Substituting the right sides of equations (7b), (9b) and (11) for their left sides, and recalling approximation (3d), we estimate $P(C) \approx$

$$\frac{b}{mk}\cdot\frac{\sigma(t_e-t_b)}{\sqrt{2\pi}}\cdot p_0 + \frac{bl}{mk}\cdot p_0 + \frac{2n_0h}{c}\cdot\frac{b}{mk}\left(l+\frac{\sigma(t_e-t_b)}{\sqrt{2\pi}}\right)$$

$$= \frac{bp_0}{mk}\left(l+\frac{\sigma(t_e-t_b)}{\sqrt{2\pi}}\right) + \frac{2n_0h}{c}\cdot\frac{b}{mk}\left(l+\frac{\sigma(t_e-t_b)}{\sqrt{2\pi}}\right)$$

$$= \frac{b}{mk}\cdot\left(l+\frac{\sigma(t_e-t_b)}{\sqrt{2\pi}}\right)\cdot\left(p_0+\frac{2n_0h}{c}\right) \approx P(O)\cdot\left(p_0+\frac{2n_0h}{c}\right). \tag{12a}$$

Remembering also that $n_0 \approx (p_0 \cdot k_0)/(2w)$, we see that $\frac{2n_0h}{c} \approx \frac{p_0k_0h}{cw}$; and so we can rewrite formula (12a) as

$$P(C) \approx p_0\cdot\left(1+\frac{k_0h}{cw}\right)\cdot P(O), \tag{12b}$$

or as

$$P(C) \approx p_0\cdot\left(1+\frac{k_0h}{cw}\right)\cdot\frac{b}{mk}\left(l+\frac{\sigma(t_e-t_b)}{\sqrt{2\pi}}\right). \tag{12c}$$

8.2 The parameter values used in figure 5 were chosen, in part, to show the possibility of computing $P(O)$ even when some of the arguments of $\Psi$ are less than 6. However, in most cases we can expect $a_2$'s vertical speed to be significantly greater than 200 feet per minute. Figure 7 shows a spreadsheet in which the parameter values are the same as those of figure 5, except that the vertical speed, $c$, is set to 400 feet per minute. The spreadsheet computes collision probability, which, in this case, is $5.15\cdot 10^{-7}$. It is the sum of a nose-to-tail-collision probability of $3.37\cdot 10^{-7}$, a top-to-bottom-collision probability of $1.34\cdot 10^{-7}$, and a side-to-side-collision probability of $0.44\cdot 10^{-7}$.

# Figure 7: Probability of Collision

| | | | | | |
|---|---|---|---|---|---|
| Probability of collision during an in-trail procedure (ITP) | | | | | |
| | | | | | |
| 1 meter = | 3.280839895 | feet | | | |
| 1 nmi = | 6076.115486 | feet | | | |
| $a$ = | 1000 | feet = | 0.164579 | nmi | |
| $h$ = | 65 | feet = | 0.010698 | nmi | |
| $c$ = | 400 | ft/min = | 3.949892 | kt | |
| $t_b$ = | $(a-h)/c$ | hr = | 0.038958 | hr | |
| $t_e$ = | $(a+h)/c$ | hr = | 0.044375 | hr | |
| $l$ = | 0.03 | nmi | | | |
| $m$ = | 30 | nmi | | | |
| $k$ = | 0.5 | | | | |
| $b$ = | 0.0001 | | | | |
| $\sigma$ = | 35 | kt | | | |
| | | | | | |
| $\sigma t_b$ = | 1.363542 | | lateral overlap probability (as in SASP-WG/WHL /11-WP/5, with $S$ = 0): | | |
| $(m+l)/(\sigma t_b)$ = | 22.023529 | | $w$ = | 0.032 | nmi |
| $\Psi[(m+l)/(\sigma t_b)]$ = | 22.023529 | | $\sigma_L$ = | 0.0232 | nmi |
| $(km+l)/(\sigma t_b)$ = | 11.022765 | | $\lambda$ = | 0.038 | nmi |
| $\Psi[(km+l)/(\sigma t_b)]$ = | 11.022765 | | $\alpha$ = | 0.00564 | |
| $\sigma t_e$ = | 1.553125 | | $\alpha(1-\alpha)$ = | 0.005608 | |
| $(m-l)/(\sigma t_e)$ = | 19.296579 | | $\exp(\sigma_L^2/2\lambda^2)$ = | 1.204869 | |
| $\Psi[(m-l)/(\sigma t_e)]$ = | 19.296579 | | $w/\lambda$ = | 0.842105 | |
| $(km-l)/(\sigma t_e)$ = | 9.638632 | | $w/\sigma_L$ = | 1.379310 | |
| $\Psi[(km-l)/(\sigma t_e)]$ = | 9.638632 | | $\sigma_L/\lambda$ = | 0.610526 | |
| $(b-k)/(mk(1-k))$ = | -0.066653 | | core-core term: | 0.663055 | |
| $(1-b)/(m(1-k))$ = | 0.06666 | | tail-tail term: | 0.000012 | |
| $b/(mk)$ = | 0.000007 | | core-tail term: | 0.005530 | |
| $\Psi(0) = 1/\sqrt{(2\pi)}$ = | 0.398942 | | $p_0$ = | 0.668598 | |
| $P(O)$ = | 7.04e-07 | (formula 3c) | | | |
| | | | relative cross-track speed for pairs of GPS airplanes (as in SASP WG/A/1-WP/6): | | |
| $P(O) \approx$ | 7.04e-07 | (formula 3d) | $s_{GPS}$ = | 0.7838 | |
| | | | $\sqrt{2}$ = | 1.414214 | |
| | | | | | |
| $P(C_{NT}) \approx$ | 3.37e-07 | (formula 7b) | $k_0 = \sqrt{2} \cdot s_{GPS}$ = | 1.108461 | |
| $P(C_{TB}) \approx$ | 1.34e-07 | (formula 9b) | | | |
| $P(C_{SS}) \approx$ | 4.42e-08 | (formula 11) | $n_0$ = | 11.579909 | |
| $P(C) \approx$ | 5.15e-07 | | | | |
| | | | | | |
| $P(C) \approx$ | 5.15e-07 | (formula 12a) | | | |

8.3 Figure 8 shows the effect of varying the parameters $k$ and $b$. In particular, we consider four possible values of $mk$, the minimum acceptable longitudinal separation at the moment when the climb or descent begins: 15 nmi, 18 nmi, 21 nmi, and 24 nmi. For each of those values we graph collision probability as a function of $b$. The figure shows that collision probability varies linearly with $b$, which is exactly what we expect in view of approximation (12c).

8.4 Though $U$ may assume a value in $(-mk, mk)$ for a variety of reasons – such as equipment failure or meteorological aberration – we expect the event $\{-mk < U < mk\}$ to result largely from human error, i.e., from "blunder". Since the values of $b$ are expected to be small numbers, it then makes sense to think of the probability $b$ as closely approximating the "blunder rate". Viewed in this way figure 8 shows the probability of collision for a range of blunder rates extending from one blunder per million executions of the ITP to one blunder per hundred executions.

## 9 Collision rate

9.1 Collision risk is traditionally expressed in units of fatal accidents per flight hour, and is compared to a target level of safety (TLS), which is a maximum acceptable estimated rate of fatal accidents per flight hour. The operation discussed in this report involves planned separation in the longitudinal dimension, but only random separation in the lateral dimension, and no separation in the vertical dimension. Thus the appropriate metric for the present discussion is more completely stated as "fatal accidents per flight hour (that are) due to the loss of planned longitudinal separation".

9.2 Suppose that for some given airspace there already exists an estimate, $r$, of the rate of fatal accidents due to the loss of planned longitudinal separation, but that the estimate does not account for the effect of climbs or descents such as those considered in this report. Also suppose that during some very long period of time – $H$ hours (presumably on the order of several decades) – the airspace can be expected to experience $n$ such climb or descent operations. Then during that time period we would expect such operations to give rise to $n \cdot \mathrm{P}(C)$ collisions, or $2n\mathrm{P}(C)$ accidents. During that period of time the airspace would have some average number $f$ of active flights, and would, therefore, generate $f$ flight-hours per hour, or $Hf$ flight-hours in $H$ hours. So, during the $H$-hour-long time period, the airspace would experience $Hfr$ accidents that were not due to ITP climb or descent operations, and $2n\mathrm{P}(C)$ accidents that were due to such operations. The total number of accidents due to the loss of planned longitudinal separation would be $Hfr + 2n\mathrm{P}(C)$; and the accident rate, in traditional units, would be $[Hfr + 2n\mathrm{P}(C)]/(Hf) = r + 2 \cdot (n/H) \cdot [\mathrm{P}(C)/f]$ accidents per flight-hour. From this last expression it's clear that the accident rate due to ITP climb or descent operations would be $2 \cdot (n/H) \cdot [\mathrm{P}(C)/f]$. We also note that since the quotient $n/H$ is the hourly rate of ITP climb or descent operations, the computed rate would be independent of the choice of time period – as long as the period weren't so short that we'd be unable to obtain a stable estimate of $n/H$.

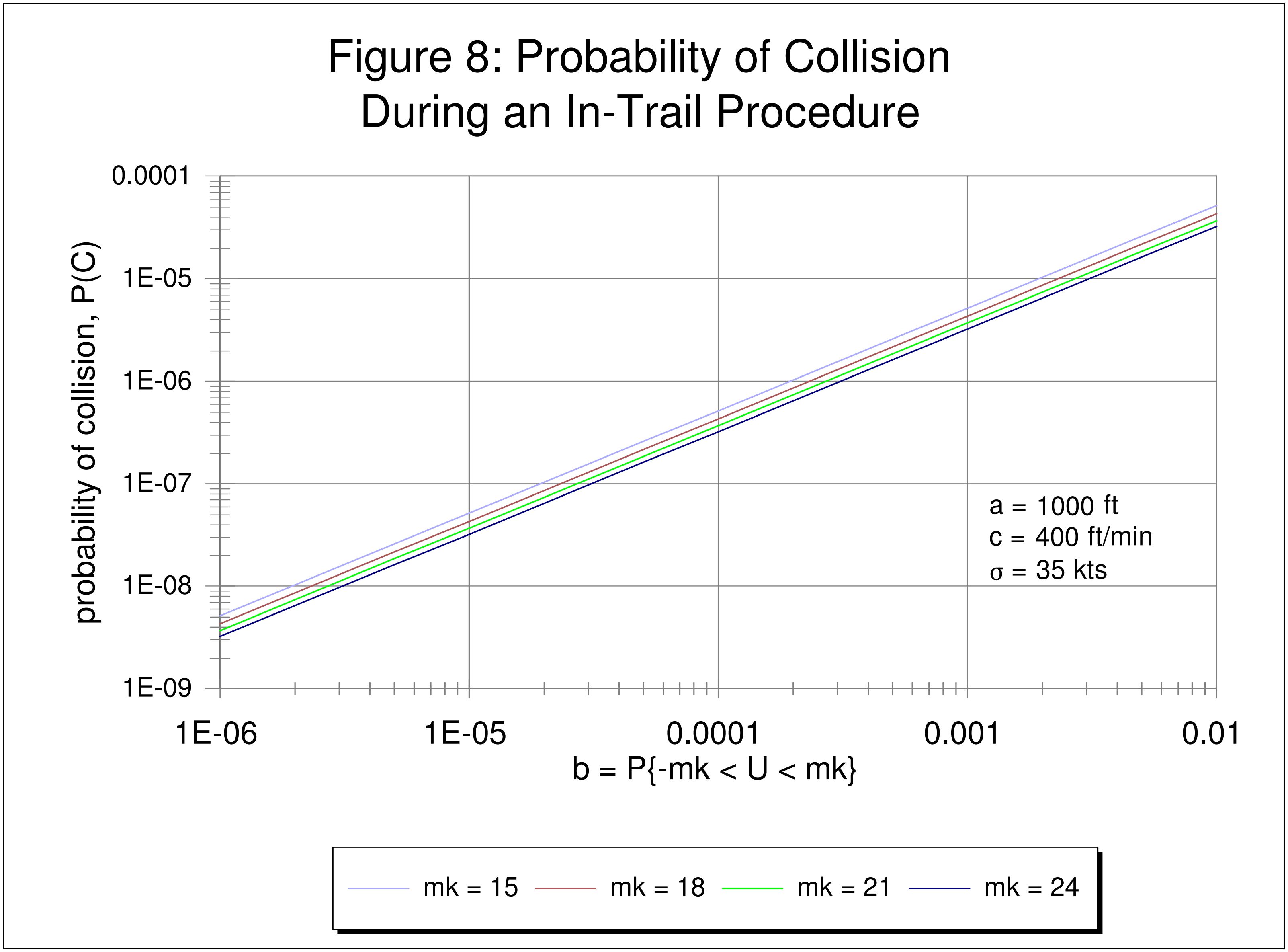


Figure 8: Probability of Collision
During an In-Trail Procedure
probability of collision, P(C)
0.0001
1E-05
1E-06
1E-07
1E-08
1E-09
1E-06
1E-05
0.0001
0.001
0.01
b = P{-mk < U < mk}
a = 1000 ft
c = 400 ft/min
σ = 35 kts
mk = 15
mk = 18
mk = 21
mk = 24

9.3 Suppose, for example, that an oceanic airspace experiences an average instantaneous traffic load of $f = 30$ flights. Suppose also that the air navigation service provider (ANSP) that operates the airspace restricts the use of the ITP to pairs of airplanes that both navigate by using the GNSS, and that such airplanes' lateral performance is similar to that of the airplanes studied in reference 10.3. The spreadsheet shown in figure 7 uses parameter values taken from that reference, and is, therefore, relevant to this example. The following table shows the numerical values plotted in figure 8, which were obtained by re-computing the spreadsheet of figure 7 with four different values of $k$ and nine different values of $b$.

Probability of Collision, P($C$), During the Execution of an ITP

| | *mk* = minimum initial longitudinal separation (nmi) | | | |
|---|---|---|---|---|
| *b* | 15 | 18 | 21 | 24 |
| 1e-02 | 5.150e-05 | 4.292e-05 | 3.679e-05 | 3.219e-05 |
| 5e-03 | 2.575e-05 | 2.146e-05 | 1.839e-05 | 1.609e-05 |
| 1e-03 | 5.150e-06 | 4.292e-06 | 3.679e-06 | 3.219e-06 |
| 5e-04 | 2.575e-06 | 2.146e-06 | 1.839e-06 | 1.609e-06 |
| 1e-04 | 5.150e-07 | 4.292e-07 | 3.679e-07 | 3.219e-07 |
| 5e-05 | 2.575e-07 | 2.146e-07 | 1.839e-07 | 1.609e-07 |
| 1e-05 | 5.150e-08 | 4.292e-08 | 3.679e-08 | 3.219e-08 |
| 5e-06 | 2.575e-08 | 2.146e-08 | 1.839e-08 | 1.609e-08 |
| 1e-06 | 5.150e-09 | 4.292e-09 | 3.679e-09 | 3.219e-09 |

Finally, suppose that $r$, the airspace's estimated rate of accidents due to the loss of planned longitudinal separation, is less than the target level of safety (TLS) $T$. Then the airspace's "budget" for accidents due to ITPs is $T - r$; and the ANSP can allow the use of ITPs as long as $T - r \geq 2 \cdot (n/H) \cdot [\mathrm{P}(C)/f]$. If the TLS is $5 \cdot 10^{-9}$ accidents per flight-hour, and $r$ is $3.5 \cdot 10^{-9}$ accidents per flight-hour, then use of the ITP could be allowed as long as $1.5 \cdot 10^{-9} \geq 2\,(n/H) \cdot \mathrm{P}(C)/30$, or, equivalently, as long as $2.25 \cdot 10^{-8} \geq (n/H) \cdot \mathrm{P}(C)$. Having estimated the value of $b$ – e.g., through a hazard analysis, or through observation of the fleet's performance during a period in which ITPs were authorized (whether in an operational trial or in normal operation) – the ANSP will be able to estimate the rate at which the procedure can be tolerated. For example, if $b = 5 \cdot 10^{-6}$, and the procedure is to be used with a minimum initial longitudinal separation of 15 nmi, then (as is shown in the table) $\mathrm{P}(C) = 2.575 \cdot 10^{-8}$, and the airspace will meet its TLS as long as $n/H \leq (2.25 \cdot 10^{-8})/(2.575 \cdot 10^{-8}) \approx 0.874$. In this example the ANSP can safely use ITPs as long as the rate of utilization is limited to approximately seven times per eight hours, or 21 times per day.

10 References

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